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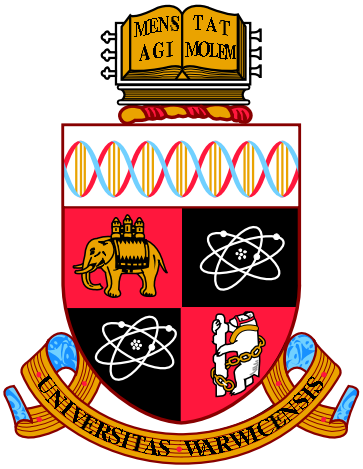

One-Dimensional Interacting Particle Systems as Pfaffian Point Processes

by

Barnaby James Garrod

Thesis

Submitted to the University of Warwick

for the degree of

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Contents

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Declarations

Sections 3.2 and 4.1, containing the construction and analysis of the coalescing and annihilating random walk (CARW) model, and the convergence lemma for Pfaffian point processes, build on the preprint [25], co-authored with my supervisors, Roger Tribe and Oleg Zaboronski, and Mihail Poplavskyi.

I declare that to the best of my knowledge, the material contained in this thesis is original and my own work except where otherwise indicated. This thesis has not been submitted in any previous application for any degree.

Abstract

A wide class of one-dimensional continuous-time discrete-space interacting particle systems are shown to be Pfaffian point processes at fixed times with kernels characterised by the solutions to associated two-dimensional ODEs. The models comprise instantaneously coalescing or annihilating random walks with fully spatially inhomogeneous jump rates and deterministic initial conditions, including additional pairwise immigration or branching in the pure interaction regimes. We formulate convergence of Pfaffian point processes via their kernels, enabling investigation of diffusive scaling limits, which boils down uniform convergence of lattice approximations to two-dimensional PDEs. Convergence to continuum point processes is developed for a subset of the discrete models. Finally, in the case of annihilating random walks with pairwise immigration we extend the picture to multiple times, establishing the extended Pfaffian property for the temporal process.

Glossary of terms

ARW	Annihilating random walks, 37
ARWI	Annihilating random walks with pairwise immigration, 50
BCRW	Branching coalescing random walks, 55
CARW	Coalescing and annihilating random walks, 37
CRW	Coalescing random walks, 37
$C_0(\Lambda)$	Space of continuous $f : \Lambda \to [0, \infty)$ with compact support, 10
$(\eta_t : t \geq 0)$	Continuous-time Markov process on $\{0,1\}^{\mathbb{Z}}$ (also $(X_t : t \geq 0)$ when scaling), 34
GOE	Gaussian orthogonal ensemble, 24
GSE	Gaussian symplectic ensemble, 24
GUE	Gaussian unitary ensemble, 24
$\mathcal{M}_{\mathrm{LFP}}(\Lambda)$	Space of locally finite point measures on Λ, 10
$\mathcal{M}_0(\Lambda)$	Space of simple measures on Λ, 10
V_{2n}	Weyl chamber $x_1 < \cdots < x_{2n}$ in $\mathbb{Z}^{2n}$, 42

Chapter 1

Overview

Identifying underlying algebraic structure of an interacting particle system unlocks its analysis. The most natural place to look is in the intensity functions, which essentially represent the probability of finding particles at given space-time positions. These functions typically characterise the process and are a pathway to investigate further properties. By algebraic structure we mean that the intensity is given by first generating, via some kernel, a square matrix indexed by the positions under consideration, and then substituting into a polynomial in the matrix entries. The polynomial of interest here is the Pfaffian, whose square is given by the determinant. There are many examples of systems governed by Pfaffians and determinants, most notably arising in random matrix theory.

The primary motivation for the models considered in this thesis comes from a system of interacting Brownian motions. In [59] Tribe and Zaboronski show that at an arbitrary fixed time the positions of instantly coalescing Brownian motions and instantly annihilating Brownian motions, started from a maximal initial condition, form Pfaffian point processes on $\mathbb{R}$. The aim is to investigate how deep this Pfaffian structure goes. We illuminate the picture by generalising in a number of ways. Firstly we establish that continuous-time random walk models on $\mathbb{Z}$ with general inhomogeneous jump rates and mixed interactions, started from deterministic initial conditions, are Pfaffian. We then extend the pure interaction models further. For the case of pure annihilation we include inhomogeneous pairwise immigration of particles, and for pure coalescence we allow inhomogeneous left and right branching of particles. Both extensions are also Pfaffian. Thus, the structure in [59] is not a coincidence, Pfaffians are fundamentally interwoven with annihilating and coalescing particle systems.

To come full circle, we develop the convergence of Pfaffian point processes via their kernels, then apply this to investigate diffusive scaling limits of the discrete point processes. We focus on convergence of spatially homogeneous models with certain initial conditions, in particular recovering the Pfaffian point processes for interacting Brownian motions.

Tribe and Zaboronski note in [59] that the one-dimensional distributions of the annihilating Brownian motion system, appropriately rescaled with time, coincide with the distribution of real eigenvalues in the bulk scaling limit of the real Ginibre ensemble of random matrix theory [8]. The eigenvalue point process under the edge scaling limit is also Pfaffian [8]. We locate this point process as the scaling limit of annihilating random walks with a one-sided initial condition, thus extending the mysterious link between annihilating Brownian motions and the real eigenvalues of the real Ginibre ensemble to the edge of the spectrum.

In [58] Tribe and Zaboronski extend their single-time result of [59] to show that the finite-dimensional marginals of instantly annihilating Brownian motions under a maximal entrance law are described by Pfaffians, identifying the model as an extended Pfaffian point process. We generalise this by proving that the annihilating random walk models with pairwise immigration satisfy the extended Pfaffian property.

1.1 Outline of the thesis

The thesis is separated into three main parts: discrete-space models at single times; the passage to and study of continuum models at single times; and multi-time analysis for discrete-space models. All of the processes considered in the thesis are in continuous time and all interactions are instantaneous. We give an outline of the content in each chapter.

We begin by introducing the foundational concept of the Pfaffian in chapter 2. The Pfaffian of a matrix is defined and basic theory reviewed, including some useful formulae for its analysis. We then turn to point processes, recapping the general framework before defining Pfaffian point processes and exploring their properties. The theory is put into action by discussing some explicit examples.

With the fundamental machinery in hand, we initiate our study in chapter 3 with discrete-space interacting particle systems. The class of models we consider com-

prises independent random walks on $\mathbb{Z}$ defined by general inhomogeneous jump rates and instantaneous coalescence or annihilation, determined at each coincidence by an independent Bernoulli variable. We show that for deterministic initial conditions the distribution of particles at an arbitrary fixed time is a Pfaffian point process with kernel characterised by the solution to an associated two-dimensional ODE. We then extend the models in the pure interaction regimes. For pure annihilation we include inhomogeneous pairwise immigration of particles, and for pure coalescence we allow inhomogeneous left and right branching of particles. Both extensions are shown to be Pfaffian with corresponding ODEs governing the kernels. We close the chapter by deriving a relation between the two extensions.

The passage to continuum point processes via scaling limits is addressed in chapter 4. We firstly develop convergence of Pfaffian point processes via their kernels. This facilitates the investigation of diffusive scaling limits for the discrete point processes, which boils down to uniform convergence of the characterising lattice ODEs to continuum PDEs. For the coalescing and annihilating random walk model, we prove convergence of the one-dimensional marginals of the scaled discrete processes in the case of symmetric homogeneous rates and independent initial conditions. The appropriate scaling for the immigration and branching models is discussed and the limit point processes identified. The continuum limits are Pfaffian point processes with kernels characterised by the associated two-dimensional limit PDEs.

Multi-time generalisations are considered in chapter 5. We introduce the notion of an extended Pfaffian point process and prove that the annihilating random walk model with pairwise immigration is an example.

Finally, some auxiliary PDE results are contained in the appendices.

Chapter 2

Introduction to Pfaffians

Pfaffians and Pfaffian point processes are central concepts of this work and in this chapter we set out the general theory.

Section 2.1 contains an introduction to Pfaffians along with some useful formulae for its analysis. In section 2.2 we turn to point processes, recapping the general framework before developing the theory of Pfaffian point processes, including properties and examples.

2.1 Pfaffian of a matrix

The concept of the Pfaffian is inextricably connected to anti-symmetry and we begin with the definition of an anti-symmetric matrix. Throughout the thesis we deal exclusively with real matrices and so restrict attention accordingly, but remark that one may just as well consider complex entries.

Definition 1. *A real square matrix A is said to be* ***anti-symmetric*** *if it satisfies*

$$A^T = -A,$$

where A^T denotes the transpose of A.

The determinant of a $2n \times 2n$ anti-symmetric matrix is the square of a polynomial in the matrix entries. This polynomial is the Pfaffian of the matrix. Its study was instigated by Pfaff early in the nineteenth century and the name Pfaffian was coined by Cayley. See Knuth [35], and references therein, for more on the origins and historical development. We now give the formal definition and basic properties,

referring to Stembridge [54] for some of the proofs. Similarly to the determinant the Pfaffian may be defined combinatorially as a sum over permutations.

Definition 2. *Let $A = (a_{ij})_{i,j=1}^{2n}$ be a real anti-symmetric matrix. The **Pfaffian** of the matrix A, denoted by $\mathrm{Pf}(A)$, is defined by*

$$\mathrm{Pf}(A) = \frac{1}{2^n n!} \sum_{\sigma \in \Sigma_{2n}} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} a_{\sigma(2i-1),\sigma(2i)},$$

where Σ_{2n} is the set of permutations on $\{1, 2, \ldots, 2n\}$.

Due to anti-symmetry there are many cancellations and the sum may be replaced by a subset of permutations, so-called perfect matchings. In particular, the Pfaffian of a matrix may be expressed in terms of the upper-triangular entries.

Proposition 1. *Let $A = (a_{ij})_{i,j=1}^{2n}$ be a real anti-symmetric matrix. Then*

$$\mathrm{Pf}(A) = \sum_{\sigma \in \Sigma'_{2n}} \mathrm{sgn}(\sigma) \prod_{\ell=1}^{n} a_{i_\ell, j_\ell},$$

where Σ'_{2n} is the set of permutations on $\{1, 2, \ldots, 2n\}$ with $\sigma(2\ell-1) = i_\ell$, $\sigma(2\ell) = j_\ell$ and $i_\ell < j_\ell$ for $\ell = 1, \ldots, n$, and $i_1 < i_2 < \cdots < i_n$.

Proof of proposition 1. We obtain the claimed expression by developing the definition of $\mathrm{Pf}(A)$. Consider the summand $\mathrm{sgn}(\sigma) \prod_{i=1}^{n} a_{\sigma(2i-1),\sigma(2i)}$ for some permutation $\sigma \in \Sigma_{2n}$. Let $\sigma' \in \Sigma_{2n}$ be obtained from σ by changing a pair $(\sigma(2i-1), \sigma(2i))$ to $(\sigma(2i), \sigma(2i-1))$. This flips the sign of the product since $a_{\sigma(2i-1),\sigma(2i)} = -a_{\sigma(2i),\sigma(2i-1)}$ by anti-symmetry. However, $\mathrm{sgn}(\sigma') = -\mathrm{sgn}(\sigma)$ since σ' may be expressed as σ followed by a single transposition $\sigma(2i-1) \leftrightarrow \sigma(2i)$. All together, the summands for σ and σ' coincide. This is also true if σ' is obtained from σ by exchanging the order of two pairs, that is, changing $(\sigma(2i-1), \sigma(2i), \sigma(2i+1), \sigma(2i+2))$ to $(\sigma(2i+1), \sigma(2i+2), \sigma(2i-1), \sigma(2i))$. Indeed the product is unaffected and $\mathrm{sgn}(\sigma') = \mathrm{sgn}(\sigma)$ since σ' may be expressed as σ followed by two transpositions. Iterating this shows that the summand does not depend on the order in each pair $(\sigma(2i-1), \sigma(2i))$, nor the order of pairs $(\sigma(1), \sigma(2))$, $(\sigma(3), \sigma(4))$, $\ldots$, $(\sigma(2n-1), \sigma(2n))$. Thus, there are $2^n n!$ permutations whose summand coincides with that for σ. This partitions Σ_{2n} into $(2n)!/(2^n n!)$ equivalence classes (corresponding to perfect matchings) and it remains to pick a representative from each, since the prefactor $1/(2^n n!)$ cancels with the number of summands in each class. Let $\sigma(2\ell-1) = i_\ell$ and $\sigma(2\ell) = j_\ell$. There is a unique permutation in each equivalence class satisfying the conditions

$i_\ell < j_\ell$ and $i_1 < i_2 < \cdots < i_n$. Indeed the first condition fixes an order in each pair $(\sigma(2\ell-1), \sigma(2\ell))$ and the second fixes an ordering of the pairs. The set of chosen representatives is exactly Σ'_{2n}. □

The connection to determinants is made explicit in the following proposition.

Proposition 2. *Let A be a $2n \times 2n$ real anti-symmetric matrix. Then*

$$\det(A) = \mathrm{Pf}(A)^2.$$

Proof of proposition 2. See [54] proposition 2.2. □

The determinant of an anti-symmetric matrix of odd dimension vanishes and accordingly the Pfaffian is defined to be zero. Indeed, let A be an anti-symmetric $(2n+1) \times (2n+1)$ matrix then

$$\det(A) = \det(A^T) = \det(-A) = (-1)^{2n+1} \det(A) = -\det(A).$$

Before exploring properties of the Pfaffian we find our footing by recording the smallest non-trivial cases:

$$\mathrm{Pf}\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} = a, \qquad \mathrm{Pf}\begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix} = af - be + dc.$$

Owing to its origin the Pfaffian enjoys many analogous properties to the determinant. The following conjugation result is useful for manipulating Pfaffians, for example, we often conjugate with the matrix of an elementary row or column operation.

Proposition 3 (Conjugation)**.** *Let A and B be $2n \times 2n$ real matrices with A anti-symmetric. Then*

$$\mathrm{Pf}(B^T A B) = \det(B)\,\mathrm{Pf}(A).$$

Proof of proposition 3. See [54] proposition 2.3. □

The next formula is an expansion for the Pfaffian of the sum of two matrices.

Proposition 4 (Summation)**.** *Let A and B be $2n \times 2n$ real anti-symmetric matrices. Then*

$$\mathrm{Pf}(A+B) = \sum_U (-1)^{|U|/2} (-1)^{s(U)} \,\mathrm{Pf}(A|_U)\,\mathrm{Pf}(B|_{U^c}),$$

where the sum is over subsets $U \subseteq \{1, \ldots, 2n\}$ *with* $|U|/2 \in \{1, \ldots, n\}$, $s(U) = \sum_{j \in U} j$ *(with* $s(\emptyset) = 0$*), and* $A|_U$ *means the matrix* A *restricted to the rows and columns indexed by* U*.*

Proof of proposition 4. See [54] lemma 4.2. □

A consequence of the summation formula is a Laplace expansion for Pfaffians. We give the formula for expanding in a row, but note that by anti-symmetry it may be rewritten as a column expansion.

Proposition 5 (Row expansion)**.** *Let* $A = (a_{ij})_{i,j=1}^{2n}$ *be a real anti-symmetric matrix. Then for any* $i \in \{1, \ldots, 2n\}$

$$\mathrm{Pf}(A) = \sum_{j=1, j \neq i}^{2n} (-1)^{i+j+1+1(j<i)} a_{ij} \, \mathrm{Pf}(A^{(i,j)}),$$

where $A^{(i,j)}$ *is the* $(2n-2) \times (2n-2)$ *submatrix formed by removing the* i*-th and* j*-th rows and columns from* A*, and* $1(\cdot)$ *is the indicator function.*

Proof of proposition 5. The expression follows upon decomposing A into the sum of two $2n \times 2n$ anti-symmetric matrices, by separating the i-th row and column from A, and applying proposition 4. □

Looking ahead, we are interested in particle systems characterised by Pfaffians. Particles in these systems are indistinguishable and the matrix indices of the Pfaffians correspond to particles. From this it is already not surprising that the matrices involved are highly structured with some symmetry among the entries. We now look at different forms of structure in matrix entries and in each case develop the Pfaffian. For convenience when dealing with anti-symmetric matrices we introduce the sign function $\mathrm{sgn} : \mathbb{R} \to \{\pm 1, 0\}$ defined by $\mathrm{sgn}(z) = 1$ if $z > 0$, $\mathrm{sgn}(z) = -1$ if $z < 0$ and $\mathrm{sgn}(z) = 0$ if $z = 0$. Although strictly this is an abuse of notation with the sign of a permutation it will always be clear from context which function is being used. When working with Pfaffians it is often more convenient to indicate a matrix by its entries. In particular if $A = (a_{ij})_{i,j=1}^{2n}$ is an anti-symmetric matrix and we wish to stress the dependence on entries we will write $\mathrm{Pf}(a_{ij} : i, j \leq 2n)$ instead of $\mathrm{Pf}(A)$. Due to anti-symmetry a Pfaffian is defined by the upper triangular entries alone, so we also write $\mathrm{Pf}(a_{ij} : i < j \leq 2n)$.

For general square matrices the simplest non-trivial matrix with entries of the same form is the constant matrix consisting of all 1's. The anti-symmetric analogue

of this has values 1 in the upper triangular entries (-1 in the lower) and is denoted by $\mathbf{1}$. It is useful to compute $\mathrm{Pf}(\mathbf{1})$ since it comes up in various places, including the proof of subsequent properties.

Proposition 6. *Let* $\mathbf{1}$ *be the* $2n \times 2n$ *matrix with entries* $\mathbf{1}_{ij} = \mathrm{sgn}(j-i)$*. Then*

$$\mathrm{Pf}(\mathbf{1}) = 1.$$

Proof of proposition 6. Proceed by induction on n. For $n = 1$ the result is trivial. For $n > 1$ expanding in the top row by proposition 5 and substituting in the inductive hypothesis

$$\mathrm{Pf}(\mathbf{1}) = \sum_{j=2}^{2n} (-1)^j \,\mathrm{Pf}(\mathbf{1}^{(1,j)}) = \sum_{j=2}^{2n} (-1)^j = 1.$$

□

The following scaling relation leads to the generalised expression $\mathrm{Pf}(c\mathbf{1}) = c^n$ for $c \in \mathbb{R}$.

Proposition 7. *Let* A *be a* $2n \times 2n$ *real anti-symmetric matrix and fix a constant* $c \in \mathbb{R}$*. Then*

$$\mathrm{Pf}(cA) = c^n \,\mathrm{Pf}(A).$$

Proof of proposition 7. Appealing directly to the definition of the Pfaffian, for any permutation the factor c^n may be pulled outside the product of entries and outside the summation. What remains in the summation is $\mathrm{Pf}(A)$. □

If the entries admit a product or quotient form then the Pfaffian is explicit.

Proposition 8. *For* $a_1, \ldots, a_{2n} \in \mathbb{R}$

$$\mathrm{Pf}(a_i a_j : i < j \leq 2n) = \prod_{i=1}^{2n} a_i.$$

Proof of proposition 8. Apply the conjugation formula to $B^T \mathbf{1} B$, where B is a diagonal matrix with entries $a_1, \ldots, a_{2n}$, and conclude with proposition 6. □

Proposition 9. *For non-zero* $a_1, \ldots, a_{2n} \in \mathbb{R}$

$$\mathrm{Pf}\left(\frac{a_i}{a_j} : i < j \leq 2n\right) = \prod_{i=1}^{n} \frac{a_{2i-1}}{a_{2i}}.$$

Proof of proposition 9. Let $A = (a_{ij})_{i,j=1}^{2n}$ denote the anti-symmetric matrix defined by its upper triangular entries $a_{ij} = a_i/a_j$ (for $i < j$). By conjugating with a suitable elementary matrix we may subtract a multiple a_1/a_2 of the second row and column from the first row and column. This produces a new matrix $\tilde{A}$, satisfying $\mathrm{Pf}(A) = \mathrm{Pf}(\tilde{A})$ by proposition 3, which has the same entries as A but with first row $(0, a_1/a_2, 0, \ldots, 0)$ (and corresponding anti-symmetric first column). Expanding $\tilde{A}$ in the first row

$$\mathrm{Pf}(A) = \mathrm{Pf}(\tilde{A}) = \frac{a_1}{a_2}\,\mathrm{Pf}(\tilde{A}^{(1,2)}) = \frac{a_1}{a_2}\,\mathrm{Pf}(A^{(1,2)}),$$

and by induction on n we find $\mathrm{Pf}(A) = (a_1 a_3 \ldots a_{2n-1})/(a_2 a_4 \ldots a_{2n})$. □

Finally, the Pfaffian factorises if a matrix is of block diagonal form.

Proposition 10. *Let A and B be real anti-symmetric matrices with dimensions $2k \times 2k$ and $(2n-2k) \times (2n-2k)$ respectively, for some $1 < k < n$. Let C denote the $2n \times 2n$ matrix whose top left $2k \times 2k$ block is given by A, bottom right $(2n-2k) \times (2n-2k)$ block is given by B, and remaining entries are all zero. Then*

$$\mathrm{Pf}(C) = \mathrm{Pf}(A)\,\mathrm{Pf}(B).$$

Proof of proposition 10. View C as the sum of two $2n \times 2n$ matrices, formed by adding rows and columns of zeroes to A and B, and apply proposition 4. □

2.2 Pfaffian Point processes

The fundamental objects in this thesis are Pfaffian point processes. In order to define them we first review some general theory of point processes in section 2.2.1. In later chapters we study processes on both discrete and continuous spaces, but rather than replicate theory for the different cases, section 2.2.1 is set in a general topological framework. With the foundations laid we define Pfaffian point processes and explore basic properties in section 2.2.2. The theory is put into action in section 2.2.3, where we discuss examples from random matrix theory and interacting particle systems, including the key model of coalescing and annihilating Brownian motions. Finally, in section 2.2.4 we show that the set of Pfaffian point processes is closed under particular thinning and thickening operations.

2.2.1 General theory of point processes

We review some classical theory of point processes. A comprehensive exposition may be found in Daley and Vere-Jones [10, 11], while references more specific to our context are Anderson et al. [4] section 4.2, and Johansson [31]. We also refer to Kallenberg [32] for measure theoretic considerations.

Let Λ be a locally compact, second countable Hausdorff space (think $\mathbb{Z}$ or $\mathbb{R}$). A measure μ on Λ is called *locally finite* if $\mu(B) < \infty$ for Borel $B \subset \Lambda$ that are relatively compact, that is, with compact closure (boundedness for $\mathbb{Z}$ or $\mathbb{R}$). Denote by $\mathcal{M}_{\mathrm{LFP}}(\Lambda)$ the space of locally finite point measures on Λ, that is locally finite measures μ such that $\mu(B) \in \mathbb{N}$ for relatively compact Borel $B \subset \Lambda$. The *vague topology* on $\mathcal{M}_{\mathrm{LFP}}(\Lambda)$ is generated by the projections $\mu \mapsto \mu f = \int_\Lambda f \, \mathrm{d}\mu$ for $f \in C_0(\Lambda)$, the space of continuous functions $f : \Lambda \to [0, \infty)$ with compact support, and we take the associated σ-algebra on $\mathcal{M}_{\mathrm{LFP}}(\Lambda)$. We view a point process as a random element of $\mathcal{M}_{\mathrm{LFP}}(\Lambda)$ characterised by its law.

Definition 3. *The* ***law of a point process*** *on Λ is a probability measure $\mathbb{P}$ on $\mathcal{M}_{\mathrm{LFP}}(\Lambda)$.*

As for random variables, it is convenient and standard practice to work with canonical variables and we refer to the *point process* X as a random measure with the law $\mathbb{P}$. In particular for each Borel $B \subset \Lambda$, $X(B)$ is an integer-valued random variable. Moreover if B is relatively compact then $\mu \in \mathcal{M}_{\mathrm{LFP}}(\Lambda)$ restricted to B may be written as

$$\mu|_B = \sum_{i=1}^{\mu(B)} \delta_{x_i}, \tag{2.1}$$

for some (not necessarily disjoint) $x_1, \ldots, x_{X(B)} \in \Lambda$. This elucidates the interpretation of point processes as random configurations of point masses, or particles, on Λ. We will not be interested in the case of multiple particles simultaneously occupying the same site. With this consideration in mind, we say that a measure $\mu \in \mathcal{M}_{\mathrm{LFP}}(\Lambda)$ is *simple* if $\mu(\{x\})$ is equal to 0 or 1 for each $x \in \Lambda$. The set of simple measures is denoted by $\mathcal{M}_0(\Lambda)$.

Definition 4. *A point process X is* ***simple*** *if $\mathbb{P}[X \text{ is simple}] = 1$.*

All of the point processes we consider are simple. With the definition in hand, we turn to the question of how best to describe and work with point processes. There are several approaches to this, suited to different objectives. Here we endorse joint intensities which have a natural interpretation in terms of configurations. We state

the central definition, fixing a reference measure $\nu \in \mathcal{M}_{\mathrm{LFP}}(\Lambda)$ (counting measure for $\mathbb{Z}$ and Lebesgue for $\mathbb{R}$).

Definition 5. *Let X be a simple point process on Λ. Suppose that there exist locally integrable functions $\rho^{(n)} : \Lambda^n \to [0, \infty)$ for $n \in \mathbb{N}$ such that for mutually disjoint Borel sets $B_1, \ldots, B_n$*

$$\mathbb{E}\left[X(B_1) \ldots X(B_n)\right] = \int_{B_1 \times \cdots \times B_n} \rho^{(n)}(x_1, \ldots, x_n)\, \nu(\mathrm{d}x_1) \ldots \nu(\mathrm{d}x_n).$$

*Then the function $\rho^{(n)}$ is called the n-**th (joint) intensity**.*

To digest the definition, note that Lebesgue's differentiation theorem (see [53], p. 106) gives for ν^n-almost every $(x_1, \ldots, x_n) \in \Lambda^n$ with distinct x_i

$$\rho^{(n)}(x_1, \ldots, x_n) = \lim_{\epsilon \downarrow 0} \frac{\mathbb{E}\left[X(\mathcal{B}_\epsilon(x_1)) \ldots X(\mathcal{B}_\epsilon(x_n))\right]}{\nu(\mathcal{B}_\epsilon(x_1)) \ldots \nu(\mathcal{B}_\epsilon(x_n))},$$

where $\mathcal{B}_\epsilon(x)$ is the ball of radius ϵ centred at $x \in \Lambda$. We obtain the following powerful intuition for distinct points

$$\text{“}\rho^{(n)}(x_1, \ldots, x_n)\nu(\mathrm{d}x_1) \ldots \nu(\mathrm{d}x_n) = \mathbb{P}[X \text{ has particles at } x_1, \ldots, x_n]\text{”}.$$

This is rigorous in the case of discrete Λ since the intensities may be identified pointwise by choosing singleton sets $B_i = \{x_i\}$. Indeed for a point process on $\mathbb{Z}$ with the counting reference measure

$$\rho^{(n)}(x_1, \ldots, x_n) = \mathbb{E}\left[X(x_1) \ldots X(x_n)\right] \qquad \text{for distinct } x_1, \ldots, x_n \in \mathbb{Z}, \tag{2.2}$$

and $\rho^{(n)}(x_1, \ldots, x_n)$ is equal to the probability that the point process has particles at $x_1, \ldots, x_n$. The intensities are therefore natural objects and we see how they should characterise point processes.

The intensities also give formulae for expected particle counts on overlapping sets. Note that the intensities are not (pointwise) unique. Indeed according to definition 5, $\rho^{(n)}$ is only defined ν^n-almost everywhere. For discrete spaces the diagonals, where $x_i = x_j$ for some $i \neq j$, are not null sets and we must define the intensity on them to work with overlapping sets. To facilitate the following proposition, we set $\rho^{(n)}(x_1, \ldots, x_n) = 0$ on the diagonals. Note that since we only consider simple point processes, this is also a natural choice to make. We introduce the falling factorial notation $\lfloor z \rfloor_k = z!/(z-k)! = z(z-1)\ldots(z-k+1)$ for $z, k \in \mathbb{N}$. Extending this to

$\mathcal{M}_{\mathrm{LFP}}(\Lambda)$, for relatively compact, mutually disjoint $B_1, \ldots, B_m$ and $n_1, \ldots, n_m \in \mathbb{N}$ satisfying $n_1 + \cdots + n_m = n$ the *(joint) falling factorial moment* is defined by

$$M_n(B_1^{n_1}, \ldots, B_m^{n_m}) = \mathbb{E}\left[\prod_{i=1}^{m} \lfloor X(B_i) \rfloor_{n_i}\right]. \tag{2.3}$$

Proposition 11. *Let $B_1, \ldots, B_m \subset \Lambda$ be relatively compact and mutually disjoint. For $n_1, \ldots, n_m \in \mathbb{N}$ satisfying $n_1 + \cdots + n_m = n$*

$$M_n(B_1^{n_1}, \ldots, B_m^{n_m}) = \int_{B_1^{n_1} \times \cdots \times B_m^{n_m}} \rho^{(n)}(x_1, \ldots, x_n)\, \nu(\mathrm{d}x_1) \ldots \nu(\mathrm{d}x_n).$$

Proof of proposition 11. See [4] lemma 4.2.5. □

We will not need it, but we remark that there exists a formula for the integral of the intensity over a general relatively compact set $B \subset \Lambda^n$ involving an associated process of ordered samples of points (see [4] lemma 4.2.5).

Taking stock, we arrive at the following understanding of intensities. A natural measure on the product space Λ^n is the *(rectangular) product measure* M_n' defined, for Borel $B_1, \ldots, B_n \subset \Lambda$, by

$$M_n'(B_1, \ldots, B_n) = \mathbb{E}\left[X(B_1) \ldots X(B_n)\right].$$

If it exists, the intensity $\rho^{(n)}$ is the Radon-Nikodym derivative of the absolutely continuous part of M_n' with respect to the product reference measure ν^n. As exemplified by proposition 11, in the context of intensities there is a more natural measure on Λ^n. It can be shown that (2.3) defines a measure M_n on Λ^n. If it exists, the intensity $\rho^{(n)}$ is the Radon-Nikodym derivative of M_n with respect to ν^n.

The questions are now when do intensities exist and when do they determine a point process. There are examples of point processes for which intensities do not exist, but these are generally pathological and all processes we consider admit intensities. Assuming that intensities exist it is often the case that they uniquely determine the point process. This is closely related to well-posedness of the classical moment problem and in particular growth conditions on the intensities. For Pfaffian point processes we derive a sufficient condition (proposition 13) for intensities to determine the process.

To summarise, intensities are a useful concept when working with point processes and they are the central object of study in this thesis.

2.2.2 Definition and properties

With the general framework set we define Pfaffian point processes and explore their properties. We bring forward the standing assumptions of section 2.2.1 and, as before, Λ can be thought of as $\mathbb{Z}$ or $\mathbb{R}$.

A point process is determined to be Pfaffian if the intensities take a certain Pfaffian form. Note that intensities take any integer number of arguments, but Pfaffians are only non-trivial for square matrices of even dimension. The two concepts are connected by assigning to each argument of the intensity a pair of rows and columns. This leads to a key object, the Pfaffian kernel, a 2×2 matrix-valued function. Note also that the intensity $\rho^{(n)}$ of a point process on Λ is only defined ν^n-almost everywhere, and this is reflected in the definition of the kernel. We denote by $L^1_{\mathrm{loc}}(\Lambda^n)$ the set of (equivalence classes of) ν^n-locally integrable functions on Λ^n taking values in $\mathbb{R}$, noting that a function $f \in L^1_{\mathrm{loc}}(\Lambda^n)$ is defined ν^n-almost everywhere.

Definition 6. *A **(Pfaffian) kernel** on Λ is a 2×2 matrix-valued function $\mathbf{K} : \Lambda^2 \to \mathbb{R}^{2\times 2}$ with $\mathbf{K}(x,y) = \begin{pmatrix} \mathbf{K}_{11}(x,y) & \mathbf{K}_{12}(x,y) \\ \mathbf{K}_{21}(x,y) & \mathbf{K}_{22}(x,y) \end{pmatrix}$, constructed by defining functions*

$$\mathbf{K}_{ij}(x,y) \in L^1_{\mathrm{loc}}(\Lambda^2), \tag{2.4a}$$

$$\mathbf{K}_{12}(x,x) \in L^1_{\mathrm{loc}}(\Lambda), \tag{2.4b}$$

satisfying the anti-symmetry condition $\mathbf{K}_{ij}(x,y) = -\mathbf{K}_{ji}(y,x)$ for $x, y \in \Lambda$.

Definition 7. *Let X be a simple point process on Λ with intensities $\{\rho^{(n)} : n \in \mathbb{N}\}$. Suppose that there exists a kernel $\mathbf{K}$ such that*

$$\rho^{(n)}(x_1, \ldots, x_n) = \mathrm{Pf}(\mathbf{K}(x_i, x_j) : i, j \leq n) \qquad \text{for } x_1, \ldots, x_n \in \Lambda, \tag{2.5}$$

*Then X is called a **Pfaffian point process** on Λ with kernel $\mathbf{K}$.*

We make some remarks on the definitions before exploring properties and examples. Firstly, the notation $\mathrm{Pf}(\mathbf{K}(x_i, x_j) : i, j \leq n)$ represents the Pfaffian of the $2n \times 2n$ matrix generated by the n^2 kernel blocks $\mathbf{K}(x_i, x_j)$ where the arguments range over $x_1, \ldots, x_n$. The anti-symmetry condition on the kernel guarantees that the matrix is anti-symmetric. Note that the ordering of the x_i is arbitrary. Indeed the matrix with the x_i and x_j entries swapped is obtained by exchanging two pairs of rows and

columns. This transformation may be realised by conjugation with a matrix of unit determinant, which by proposition 3 does not alter the Pfaffian. For convenience we generally (relabelling if necessary) assume the ordering $x_1 < x_2 < \cdots < x_n$. We introduce the notation $\mathbf{K}(\mathbf{x})$ for the matrix of (2.5) where $\mathbf{x} = (x_1, \ldots, x_n)$, and in the case $x_1 < \cdots < x_n$ when stressing dependence on upper triangular entries we write $\mathrm{Pf}(\mathbf{K}(x_i, x_j) : i < j \leq n)$ for $\mathrm{Pf}(\mathbf{K}(\mathbf{x}))$. Although (2.5) is a ν^n-almost everywhere equality, the definition satisfies the desired condition $\rho^{(n)}(x_1, \ldots, x_n) = 0$ if $x_i = x_j$ for some $i \neq j$. Indeed, in this case the pairs of rows (and columns) of the matrix corresponding to x_i and x_j are identical. By proposition 3, subtracting one of the rows (and corresponding column) from its counterpart by conjugating with an elementary matrix leaves the Pfaffian unchanged, but results in a row of zeroes. Expanding the Pfaffian in this row shows that it indeed vanishes. Note that the law of a Pfaffian point process depends on the kernel on the diagonal, for example $\mathbb{E}[X(B)] = \int_B \mathbf{K}_{12}(x, x)\, \nu(\mathrm{d}x)$. In fact, expanding the definition of intensities the terms of $\mathbb{E}[X(B_1) \ldots X(B_n)]$ involving the diagonal are all, by anti-symmetry, of the form $\int_{B_i} \mathbf{K}_{12}(x, x)\, \nu(\mathrm{d}x)$. In the case $\Lambda = \mathbb{R}$, however, $\{x = y\} \subset \Lambda^2$ is a ν^2-null set and so $\mathbf{K}_{12}(x, x)$ is not defined by (2.4a) alone. This is the reason we allow a pointwise description (2.4b) of $\mathbf{K}_{12}$ (up to ν-null sets). Note that in the discrete case $\Lambda = \mathbb{Z}$ the only ν^n-null set of Λ^n is the empty set and so (2.4b) is superfluous.

Remark 1. There is an alternative definition of a continuum Pfaffian point process in terms of integral operators. The motivation comes from the well-established theory of determinantal point processes, which we briefly review, referring to [51] for a full account. First studied by Macchi [40], a point process X on $\mathbb{R}$ is called determinantal if its intensities are of the form $\rho^{(n)}(x_1, \ldots, x_n) = \det(K(x_i, x_j) : i, j \leq n)$ for a kernel $K : \mathbb{R}^2 \to [0, \infty)^2$. In order to give a classification of determinantal point processes we consider kernels K that act as the kernel for a non-negative locally trace class integral operator $\mathcal{K}$ on $L^2(\mathbb{R})$. More precisely, the integral operator $\mathcal{K} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ associated to a kernel $K : \mathbb{R}^2 \to [0, \infty)^2$ is defined by

$$(\mathcal{K}f)(x) = \int_{\mathbb{R}} K(x, y) f(y)\, \mathrm{d}y \qquad \text{for } x \in \mathbb{R},\ f \in L^2(\mathbb{R}).$$

Let e_1, e_2, ... be an orthonormal basis of $L^2(\mathbb{R})$ with inner product given by $\langle f, g \rangle = \int_{\mathbb{R}} f(y) g(y)\, \mathrm{d}y$ for f, $g \in L^2(\mathbb{R})$, then the operator $\mathcal{K}$ is non-negative if $\langle \mathcal{K}f, f \rangle \geq 0$ for all $f \in L^2(\mathbb{R})$ and of trace class if

$$\mathrm{Tr}(\mathcal{K}) = \sum_{\ell=1}^{\infty} \langle \mathcal{K} e_\ell, e_\ell \rangle < \infty.$$

The trace is also given by summing the eigenvalues of $\mathcal{K}$. Finally, $\mathcal{K}$ is locally trace class if the restricted operator $1_B\mathcal{K}1_B$, with kernel $1_B(x)K(x,y)1_B(y)$, is trace class for bounded Borel $B \subset \mathbb{R}$, where 1_B is the indicator of the set B. The operator $\mathcal{K}$ characterises the kernel $K(x,y)$ up to Lebesgue null sets of $\mathbb{R}^2$, but note that the law of the point process X depends on the diagonal values $K(x,x)$, for example $\mathbb{E}[X(B)] = \int_B K(x,x)\,\mathrm{d}x$. Expanding the definition of intensities the terms of $\mathbb{E}[X(B_1)\dots X(B_n)]$ involving $K(x,x)$ are all of the form $\int_{B_i} K(x,x)\,\mathrm{d}x$, and it suffices to add the constraint

$$\mathrm{Tr}\,(1_B\mathcal{K}1_B) = \int_B K(x,x)\,\mathrm{d}x \qquad \text{for bounded Borel } B \subset \mathbb{R}.$$

For non-negative locally trace class operators $\mathcal{K}$ it is indeed possible to pick such a kernel K. Under this choice of kernel there is a simple characterisation result: a Hermitian locally trace class operator $\mathcal{K}$ determines a determinantal point process if and only if $0 \leq \mathcal{K} \leq 1$, in the sense that both $\mathcal{K}$ and $1-\mathcal{K}$ are non-negative operators. Returning to Pfaffians, the alternative definition of a Pfaffian point process on $\mathbb{R}$ (see Soshnikov [52], for instance) is that (2.5) holds for a kernel $\mathbf{K}(x,y) = \left(\begin{smallmatrix} \mathbf{K}_{11}(x,y) & \mathbf{K}_{12}(x,y) \\ \mathbf{K}_{21}(x,y) & \mathbf{K}_{22}(x,y) \end{smallmatrix}\right)$ which acts as a kernel for a locally trace class integral operator $\mathcal{K}$ on $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$. In particular, the integral operator $\mathcal{K} : L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \to L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ associated to a kernel $\mathbf{K}$ is defined, for $f = (f_1, f_2)$ with $f_1,\ f_2 \in L^2(\mathbb{R})$, by

$$\begin{aligned}(\mathcal{K}f)(x_1,x_2) &= \int_{\mathbb{R}} \begin{pmatrix} \mathbf{K}_{11}(x_1,y) & \mathbf{K}_{12}(x_1,y) \\ \mathbf{K}_{21}(x_2,y) & \mathbf{K}_{22}(x_2,y) \end{pmatrix} \begin{pmatrix} f_1(y) \\ f_2(y) \end{pmatrix} \mathrm{d}y \\ &= \begin{pmatrix} \int_{\mathbb{R}} \left(K_{11}(x_1,y)f_1(y) + K_{12}(x_1,y)f_2(y)\right)\mathrm{d}y \\ \int_{\mathbb{R}} \left(K_{21}(x_2,y)f_1(y) + K_{22}(x_2,y)f_2(y)\right)\mathrm{d}y \end{pmatrix},\end{aligned}$$

and the inner product on $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ is given, for $f = \left(f^{(1)}, f^{(2)}\right)$ and $g = \left(g^{(1)}, g^{(2)}\right)$, by

$$\langle f, g\rangle = \int_{\mathbb{R}} f^{(1)}(y)g^{(1)}(y)\,\mathrm{d}y + \int_{\mathbb{R}} f^{(2)}(y)g^{(2)}(y)\,\mathrm{d}y.$$

Comparing with definition 6, condition (2.4a) ensures that a kernel $\mathbf{K}$ acts as a kernel for an integral operator on $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$. However, there is still the issue of determining the kernel on the diagonals. By anti-symmetry it suffices to define the integrals $\int_B \mathbf{K}_{12}(x,x)\,\mathrm{d}x$ for bounded Borel $B \subset \mathbb{R}$. Instead of imposing a local trace condition we offer (2.4b). The reason is that we are not aware of a classification theorem for Pfaffian point processes in terms of integral operators

and in practice it is not straightforward to check whether a kernel defines a locally trace class operator. With this in mind, it is natural to characterise the integrals $\int_B \mathbf{K}_{12}(x,x)\,\mathrm{d}x$ by allowing a direct pointwise definition of $\mathbf{K}_{12}(x,x) : \mathbb{R} \to \mathbb{R}$ (up to null sets of $\mathbb{R}$), namely (2.4b). This is also convenient, for example Poisson processes are then Pfaffian with simple kernels and more importantly so are branching coalescing systems. In any case, this thesis is self-contained and does not rely on any external results for Pfaffian point processes, so we are justified in choosing the most convenient definition. Note that in the discrete case there is no ambiguity in the definition of a Pfaffian point process, because the intensity function is exactly the occupation probability (2.2).

The point process X in definition 7 is well-defined in the sense that its law does not depend on the choice of representatives for the equivalence classes in (2.4a) and (2.4b). Note, however, that the kernel of a Pfaffian point process is not uniquely determined. This is easy to acknowledge in light of the conjugation formula (proposition 3), for one may conjugate with a determinant one matrix but preserve the kernel form. This is the standard method for manipulating kernels and we now give some common examples.

Proposition 12. *Let X be a Pfaffian point process on Λ with kernel $\mathbf{K} = \left(\begin{smallmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{smallmatrix}\right)$. The kernel $\tilde{\mathbf{K}}$ may be taken as an alternative kernel for X in the following cases:*

1. *(Shifting constants)* $\tilde{\mathbf{K}} = \left(\begin{smallmatrix} c\mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \frac{1}{c}\mathbf{K}_{22} \end{smallmatrix}\right)$ *for $c \neq 0$;*
2. *(Inhomogeneous shift)* $\tilde{\mathbf{K}}(x,y) = \left(\begin{smallmatrix} f(x)f(y)\mathbf{K}_{11}(x,y) & \frac{f(x)}{f(y)}\mathbf{K}_{12}(x,y) \\ \frac{f(y)}{f(x)}\mathbf{K}_{21}(x,y) & \frac{1}{f(x)f(y)}\mathbf{K}_{22}(x,y) \end{smallmatrix}\right)$ *for non-zero $f : \Lambda \to \mathbb{R}$;*
3. *(Adding rows/columns)* $\tilde{\mathbf{K}} = \left(\begin{smallmatrix} \mathbf{K}_{11} & \mathbf{K}_{12}+c\mathbf{K}_{11} \\ \mathbf{K}_{21}+c\mathbf{K}_{11} & \mathbf{K}_{22}+c\mathbf{K}_{21}+c\mathbf{K}_{12}+c^2\mathbf{K}_{11} \end{smallmatrix}\right)$ *for $c \in \mathbb{R}$;*
4. *(Swapping order)* $\tilde{\mathbf{K}} = -\left(\begin{smallmatrix} \mathbf{K}_{22} & \mathbf{K}_{21} \\ \mathbf{K}_{12} & \mathbf{K}_{11} \end{smallmatrix}\right)$.

Proof of proposition 12. It suffices to show that $\tilde{\mathbf{K}}$ produces the same intensities as $\mathbf{K}$. Fix $n \in \mathbb{N}$ and $\mathbf{x} = (x_1, \ldots, x_n) \in \Lambda^n$ then we must show that $\mathrm{Pf}(\tilde{\mathbf{K}}(\mathbf{x})) = \mathrm{Pf}(\mathbf{K}(\mathbf{x}))$. To see this apply proposition 3 to $B^T\mathbf{K}(\mathbf{x})B$ with the $2n \times 2n$ determinant one matrix B given, in each case, by the elementary matrix corresponding to:

1. part 2. in the special case $f(x) = \sqrt{c}$;
2. multiplying the $2i-1$-st column by $f(x_i)$ and the $2i$-th by $f(x_i)^{-1}$ for $i = 1, \ldots, n$, namely a diagonal matrix with entries

$$\left(f(x_1), f(x_1)^{-1}, f(x_2), f(x_2)^{-1}, \ldots, f(x_n), f(x_n)^{-1}\right);$$

3. adding a multiple c of the $2i-1$-st column to the $2i$-th for $i = 1, \ldots, n$, namely a block diagonal matrix with blocks $\left(\begin{smallmatrix} 1 & c \\ 0 & 1 \end{smallmatrix}\right)$;

4. swapping the $2i-1$-st and $2i$-th columns for $i = 1, \ldots, n$, namely a block diagonal matrix with blocks $\left(\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right)$.

Note that for part 1 the matrix B may have complex entries, which is *a priori* outside the scope of part 2. However, the resulting matrix $B^T\mathbf{K}(\mathbf{x})B$ is real and proposition 3 extends directly. □

Another important property is that a bounded kernel is sufficient to determine the associated point process.

Proposition 13. *Let X be a Pfaffian point process on Λ ($\mathbb{Z}$ or $\mathbb{R}$) with kernel $\mathbf{K}$. Suppose that $\mathbf{K}$ is locally bounded in the sense that for each interval $[a, b] \subset \mathbb{R}$*

$$\|\mathbf{K}\|_{[a,b]} = \sup_{x,y\in[a,b]} \max_{i,j\in\{1,2\}} |\mathbf{K}_{ij}(x, y)| \leq C(a, b),$$

for some constant $C(a, b) > 0$ depending on a and b. Then the law of X is determined on $\mathcal{M}_{\mathrm{LFP}}(\Lambda)$.

Proof of proposition 13. The family of laws of the random variables $X(f) = \int_\Lambda f \, \mathrm{d}X$ for $f \in C_0(\Lambda)$ determine the law of X on $\mathcal{M}_{\mathrm{LFP}}(\Lambda)$ (see [32] p. 225). We show that the law of each $X(f)$ is determined by the intensities, and hence the kernel. Firstly, each $f \in C_0(\Lambda)$ may be uniformly approximated from below by step functions $f_k(x) = \sum_{i=1}^k c_i 1(x \in B_i)$ for $c_i \geq 0$ and bounded, mutually disjoint intervals $B_1, \ldots, B_k$. The moments of $X(f_k)$ are given by

$$\mathbb{E}\Big[X(f_k)^n\Big] = \sum_{m_1+\cdots+m_k=n} \binom{n}{m_1, \ldots, m_k} \prod_{i=1}^k c_i^{m_i} \mathbb{E}\Big[\prod_{i=1}^k X(B_i)^{m_i}\Big].$$

The joint moments of $X(B_i)$ may be written in terms of the falling factorial moments

$$\mathbb{E}\Big[\prod_{i=1}^k X(B_i)^{m_i}\Big] = \sum_{r_1=1}^{m_1} \cdots \sum_{r_k=1}^{m_k} \prod_{i=1}^k S(m_i, r_i) \mathbb{E}\Big[\prod_{i=1}^k \lfloor X(B_i) \rfloor_{r_i}\Big],$$

where $S(m, r)$ denotes the Stirling numbers of the second kind (see [10], for example). By proposition 11 the falling factorial moments, and hence the moments, are determined by the intensities on $B_1^{r_1} \times \cdots \times B_k^{r_k}$. In fact, proposition 11 itself, and hence the moments, only require the intensities on $B_1 \times \cdots \times B_k$, that is on bounded, mutually disjoint intervals. It remains to recover the moments of

$X(f)$ and show that the corresponding moment problem is well-posed. Note that there exist a, $b \in \Lambda$ such that $\mathbb{E}\left[X(f)^n\right] \leq M_f^n \, \mathbb{E}\big[X([a,b])^n\big]$ where $\operatorname{supp}(f) \subset [a,b]$ and $M_f = \sup_{x \in \Lambda} f(x) < \infty$. By proposition 2, Hadamard's inequality (see [29] p. 477) and the local kernel bound, the intensity at $\mathbf{x} = (x_1, \ldots, x_n)$ for $x_i \in [a,b]$ is bounded as follows

$$|\operatorname{Pf}(\mathbf{K}(\mathbf{x}))| = |\det(\mathbf{K}(\mathbf{x}))|^{1/2} \leq \|\mathbf{K}\|_{[a,b]}^n (2n)^{n/2} \leq (2C(a,b))^n n^n, \tag{2.6}$$

noting for the last inequality that $(2n)^{n/2} \leq (2n)^n$. Substituting into proposition 11, the falling factorial moments are bounded by $\mathbb{E}\left[\lfloor X([a,b]) \rfloor_n\right] \leq (2C(a,b)\nu([a,b]))^n \, n^n$. Finally, using

$$z^n \leq \begin{cases} 3^n \lfloor z \rfloor_n, & \text{if } z \geq 3n/2, \\ (3n/2)^n, & \text{if } z \leq 3n/2, \end{cases}$$

for $z \in \mathbb{R}$, the moments may be (crudely) bounded by the falling factorial moments, giving $\mathbb{E}\left[X([a,b])^n\right] \leq C'(a,b)^n n^n$ for a constant $C'(a,b)$ depending only a and b. Thus, absorbing M_f^n into the constant we arrive at the bound $\mathbb{E}\left[X(f)^n\right] \leq C'(a,b)^n n^n$. Finally, the dominated convergence theorem gives convergence of the moments $\mathbb{E}\left[X(f_k)^n\right] \to \mathbb{E}\left[X(f)^n\right]$, and the moment problem for $X(f)$ is well posed, for example the Carleman condition is satisfied (see [2], p. 65).

Alternatively, note that the bound (2.6) on the intensities is sufficient to determine the law of the point process X by a condition of Lenard [38]. □

Remark 2. For Pfaffian point processes on $\mathbb{R}$ with locally bounded kernels, definition 5 for intensities can be reduced to bounded, mutually disjoint intervals $B_1, \ldots, B_k \subset \mathbb{R}$. Indeed the proof of proposition 13 shows that the intensities on these sets determine the law of the point process.

2.2.3 Examples

We put the above theory into practice by considering some examples of Pfaffian point processes, including those arising from classical random matrix ensembles and the key motivational model of coalescing and annihilating Brownian motions.

Example 1 (Product Bernoulli on $\mathbb{Z}$)**.** Define a point process X on $\mathbb{Z}$ by setting $X(x)$ to be independent Bernoulli(λ_x) random variables for some $\lambda_x \in [0,1]$. The

intensities of X are given by

$$\rho^{(n)}(x_1, \ldots, x_n) = \mathbb{E}\left[X(x_1) \ldots X(x_n)\right] = \prod_{i=1}^{n} \lambda_{x_i}.$$

A simple Pfaffian kernel for this point process is $\mathbf{K}(x, y)$ defined by

$$\mathbf{K}(x, x) = \begin{pmatrix} 0 & \lambda_x \\ -\lambda_x & 0 \end{pmatrix}, \qquad \mathbf{K}(x, y) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{for } x \neq y.$$

By iteratively applying elementary row and column operations, we may derive alternative simple kernels, for example

$$\mathbf{K}^{(1)}(x, x) = \begin{pmatrix} 0 & \lambda_x \\ -\lambda_x & 0 \end{pmatrix}, \qquad \mathbf{K}^{(1)}(x, y) = \begin{pmatrix} \lambda_x & \lambda_x \\ \lambda_x & \lambda_x \end{pmatrix} \quad \text{for } x < y.$$

Rather than list alternative kernels, we turn to the more interesting continuum counterpart.

Example 2 (Poisson process on $\mathbb{R}$)**.** Suppose that X is a Poisson point process on $\mathbb{R}$ with intensity measure $\lambda(x)\,\mathrm{d}x$ for some $\lambda \in L^1_{\mathrm{loc}}(\mathbb{R})$ with $\lambda(x) \geq 0$. The expected particle count on disjoint sets $B_1, \ldots, B_n$ is

$$\mathbb{E}\left[X(B_1) \ldots X(B_n)\right] = \prod_{i=1}^{n} \mathbb{E}\left[X(B_i)\right] = \prod_{i=1}^{n} \int_{B_i} \lambda(x)\mathrm{d}x,$$

whereby we may take as intensity functions $\rho^{(n)}(x_1, \ldots, x_n) = \prod_{i=1}^{n} \lambda(x_i)$. A simple Pfaffian kernel for X is

$$\mathbf{K}(x, x) = \begin{pmatrix} 0 & \lambda(x) \\ -\lambda(x) & 0 \end{pmatrix}, \qquad \mathbf{K}(x, y) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{for } x \neq y.$$

Note that $\mathbf{K}$ fits definition 6, however using the alternative definition in remark 1 the integral operator defined by $\mathbf{K}$ is the zero operator. As in the discrete analogue, there are many equivalent kernels. For the homogeneous case $\lambda(x) = \lambda \in [0, \infty)$, elementary row and column operations give the equivalent kernel

$$\mathbf{K}^{(1)}(x, y) = \lambda \begin{pmatrix} \operatorname{sgn}(y - x) & 1 \\ -1 & -\operatorname{sgn}(y - x) \end{pmatrix} \quad \text{for } x, y \in \mathbb{R}.$$

A kind of spatially-dependent equivalent kernel is given by

$$\mathbf{K}^{(2)}(x,y) = -\lambda \begin{pmatrix} e^{-\lambda|y-x|}\operatorname{sgn}(y-x) & -e^{-\lambda|y-x|} \\ e^{-\lambda|y-x|} & e^{-\lambda|y-x|}\operatorname{sgn}(y-x) \end{pmatrix}. \tag{2.7}$$

Note that for both of these examples the corresponding integral operator is not the zero operator and should be locally of trace class. We give a proof that $\mathbf{K}^{(2)}$ is a kernel for the rate λ Poisson process.

Proof of kernel $\mathbf{K}^{(2)}$. It suffices to show that the intensity λ^n is given by $\operatorname{Pf}(\mathbf{K}^{(2)}(\mathbf{x}))$ for $\mathbf{x} = (x_1, \ldots, x_n)$, where without loss of generality we may assume that $x_1 < \cdots < x_n$. The approach is to introduce a matrix whose Pfaffian we can compute and then relate it to the claimed Pfaffian. Fix $y_1 < y_2 < \cdots < y_{2n}$ and define the anti-symmetric $2n \times 2n$ matrix $A = (a_{ij})$ by its upper-triangular entries

$$a_{ij} = e^{\lambda(y_i - y_j)} \qquad \text{for } i < j.$$

The entries of A are in quotient form and proposition 9 gives

$$\operatorname{Pf}(A) = \operatorname{Pf}\left(\frac{e^{\lambda y_i}}{e^{\lambda y_j}} : i < j \le 2n\right) = \prod_{i=1}^{n} e^{\lambda(y_{2i-1} - y_{2i})}. \tag{2.8}$$

To recover the desired Pfaffian we take derivatives in the even-indexed variables and then take limits as $y_{2i} \downarrow y_{2i-1}$. By definition the Pfaffian of A is a sum of products $a_{\pi_1,\pi_2} \ldots a_{\pi_{2n-1},\pi_{2n}}$, where π is a permutation on $\{1, \ldots, 2n\}$. Each product only contains each variable y_i once and the derivative $\partial_{y_{2i}}$ acts on the factor $a_{\pi_k,\pi_{k+1}}$ where either $\pi_k = 2i$ or $\pi_{k+1} = 2i$. Recombining the sum gives the Pfaffian A in which each entry involving y_{2i}, forming one row and one column, is differentiated with respect to y_{2i}. Differentiating with respect to all of the even-indexed variables gives $\partial_{y_2} \ldots \partial_{y_{2n}} \operatorname{Pf}(A) = \operatorname{Pf}(A')$, where A' splits into 2×2 blocks given on the diagonal and off the diagonal by

$$\begin{pmatrix} 0 & \partial_{y_{2i}} a_{2i-1,2i} \\ \partial_{y_{2i}} a_{2i,2i-1} & 0 \end{pmatrix}, \qquad \begin{pmatrix} a_{2i-1,2j-1} & \partial_{y_{2j}} a_{2i-1,2j} \\ \partial_{y_{2i}} a_{2i,2j-1} & \partial_{y_{2i}} \partial_{y_{2j}} a_{2i,2j} \end{pmatrix}.$$

The same method applies to the limits which act on the even-indexed variables. All together

$$\lim_{y_2 \downarrow y_1} \ldots \lim_{y_{2n} \downarrow y_{2n-1}} \partial_{y_2} \ldots \partial_{y_{2n}} \operatorname{Pf}(A) = \operatorname{Pf}(\tilde{A}'),$$

where the 2×2 blocks of $\tilde{A}'$ are given by

$$\begin{pmatrix} 0 & \lim_{y_{2i} \downarrow y_{2i-1}} \partial_{y_{2i}} a_{2i-1,2i} \\ \lim_{y_{2i} \downarrow y_{2i-1}} \partial_{y_{2i}} a_{2i,2i-1} & 0 \end{pmatrix},$$
$$\begin{pmatrix} a_{2i-1,2j-1} & \lim_{y_{2j} \downarrow y_{2j-1}} \partial_{y_{2j}} a_{2i-1,2j} \\ \lim_{y_{2i} \downarrow y_{2i-1}} \partial_{y_{2i}} a_{2i,2j-1} & \lim_{y_{2i} \downarrow y_{2i-1}} \lim_{y_{2j} \downarrow y_{2j-1}} \partial_{y_{2i}} \partial_{y_{2j}} a_{2i,2j} \end{pmatrix}.$$

Performing the operations on a_{ij} entries the diagonal and upper-triangular blocks of $\tilde{A}'$ are given by

$$\begin{pmatrix} 0 & -\lambda \\ \lambda & 0 \end{pmatrix}, \qquad \begin{pmatrix} e^{\lambda(y_{2i-1} - y_{2j-1})} & -\lambda e^{\lambda(y_{2i-1} - y_{2j-1})} \\ \lambda e^{\lambda(y_{2i-1} - y_{2j-1})} & \lambda^2 e^{\lambda(y_{2i-1} - y_{2j-1})} \end{pmatrix}.$$

We have shown that

$$\lim_{y_2 \downarrow y_1} \dots \lim_{y_{2n} \downarrow y_{2n-1}} \partial_{y_2} \dots \partial_{y_{2n}} \operatorname{Pf}(A) = \operatorname{Pf}(\tilde{\mathbf{K}}(\mathbf{y}_{\text{odd}})),$$

where $\mathbf{y}_{\text{odd}} = (y_1, y_3, \dots, y_{2n-1})$ and the kernel $\tilde{\mathbf{K}}$ is defined, for x, $y \in \mathbb{R}$, by

$$\tilde{\mathbf{K}}(x,y) = \begin{pmatrix} e^{-\lambda|y-x|} \operatorname{sgn}(y-x) & -\lambda e^{-\lambda|y-x|} \\ \lambda e^{-\lambda|y-x|} & \lambda^2 e^{-\lambda|y-x|} \operatorname{sgn}(y-x) \end{pmatrix}.$$

By proposition 12 part 1, we can move a factor of λ from $\tilde{\mathbf{K}}_{22}$ to $\tilde{\mathbf{K}}_{11}$, and then by proposition 7 introduce a factor of -1. This gives the claimed kernel $(-1)^n \operatorname{Pf}(\tilde{\mathbf{K}}(\mathbf{y}_{\text{odd}})) = \operatorname{Pf}(\mathbf{K}^{(2)}(\mathbf{y}_{\text{odd}}))$. Applying the operations to the right-hand side of (2.8) we arrive at the desired intensity

$$\operatorname{Pf}(\mathbf{K}^{(2)}(\mathbf{y}_{\text{odd}})) = (-1)^n \lim_{y_2 \downarrow y_1} \dots \lim_{y_{2n} \downarrow y_{2n-1}} \partial_{y_2} \dots \partial_{y_{2n}} \prod_{i=1}^{n} e^{2\lambda(y_{2i-1} - y_{2i})} = \lambda^n.$$

□

Example 3 (Determinantal point processes)**.** A simple point process on Λ is called *determinantal* if its intensities satisfy

$$\rho^{(n)}(x_1, \dots, x_n) = \det\left(K(x_i, x_j) : i, j \le n\right) \qquad \text{for } x_1, \dots, x_n \in \Lambda,$$

for some (*determinantal*) kernel $K : \Lambda^2 \to \mathbb{R}$. See Soshnikov [51] and remark 1 above for more details. Any determinantal point process is also Pfaffian. Indeed a determinantal point process with kernel K is equal in distribution to a Pfaffian

point process with kernel $\mathbf{K}(x,y) = \begin{pmatrix} 0 & K(x,y) \\ -K(y,x) & 0 \end{pmatrix}$, for example. Assuming that K is locally bounded, the law is determined by proposition 13 and it suffices to show that intensities coincide. This follows from a determinantal expression for Pfaffian matrices which are chequered with zeroes.

Proposition 14 (Chequerboard Pfaffian)**.** *Let $A = (a_{ij})_{i,j=1}^{2n}$ be a $2n \times 2n$ real anti-symmetric matrix satisfying $a_{ij} = 0$ if $i + j$ is even. Then*

$$\mathrm{Pf}(A) = \det\left(\tilde{A}\right),$$

where $\tilde{A} = (\tilde{a}_{ij})_{i,j=1}^{n}$ is an $n \times n$ matrix with entries $\tilde{a}_{ij} = a_{2i-1,2j}$.

Proof of proposition 14. The result follows by comparing the expansions of the Pfaffian (proposition 5) and determinant along their top rows and using an inductive argument in n. □

Random matrix ensembles. The most widely studied Pfaffian point processes are those arising in classical models of random matrix theory. A random matrix ensemble is a probability measure on a space of $N \times N$ matrices. Each ensemble has an associated point process given by the induced random eigenvalues. Over the last 50 years eigenvalue distributions and their universality properties as $N \to \infty$ have been intensely studied and the Pfaffian shown to be a fundamental structure.

Example 4 (Real Ginibre ensemble)**.** Random matrices with independent real, complex or quaternion Gaussian entries were introduced by Ginibre [26]. The most challenging of the three classical models is the real Ginibre ensemble, in which the matrix entries are independent standard Gaussian random variables. The ensemble may be defined as the following probability density, with respect to the product Lebesgue measure, on $\mathbb{R}^{N\times N}$

$$\mathrm{d}\mu_N(\mathbf{z}) = (2\pi)^{-N^2/2} e^{-\operatorname{Tr}(\mathbf{z}^T\mathbf{z})/2}\,\mathrm{d}\mathbf{z}.$$

Since the matrix entries are real, the fundamental theorem of algebra implies that almost surely the N eigenvalues are either real or come in complex conjugate pairs. Through a burst of activity, the intensity functions and their asymptotics were eventually computed by the random matrix community, see [8, 23, 48–50]. In particular, the intensities for the real eigenvalues of the real Ginibre ensemble on $N \times N$ matrices are Pfaffian. By suitably scaling with N, the eigenvalue process converges as $N \to \infty$ with different limits emerging in the bulk or near the edge of the spectrum.

One should scale so that the expected number of eigenvalues per unit interval converges. The appropriate scaling is indicated by a classical result of Ginibre [26], the circular law, stating that under the normalisation $N^{-1/2}$ the (complex) eigenvalue density converges to the uniform distribution on the unit disc as $N \to \infty$. In particular, the real eigenvalues in the bulk and edge scaling limits form Pfaffian point processes on $\mathbb{R}$ with explicit kernels. We introduce the Gaussian error function and its compliment

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-w^2} \, \mathrm{d}w, \qquad \operatorname{erfc}(z) = 1 - \operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-w^2} \, \mathrm{d}w. \tag{2.9}$$

The limit process of real eigenvalues of the real Ginibre ensemble in the bulk scaling regime as $N \to \infty$ is a Pfaffian point process on $\mathbb{R}$ with kernel given in [8] by

$$\mathbf{K}_{\text{Bulk}}^{\text{Ginibre}}(x,y) = \begin{pmatrix} \frac{1}{\sqrt{2\pi}}(y-x)e^{-\frac{1}{2}(y-x)^2} & \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(y-x)^2} \\ -\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(y-x)^2} & \frac{1}{2}\operatorname{sgn}(x-y)\operatorname{erfc}\left(\frac{|y-x|}{\sqrt{2}}\right) \end{pmatrix}.$$

Moving a factor of $-2^{-1/2}$ from $(\mathbf{K}_{\text{Bulk}}^{\text{Ginibre}})_{22}$ to $(\mathbf{K}_{\text{Bulk}}^{\text{Ginibre}})_{11}$ (by proposition 12 part 1) gives an equivalent kernel $\tilde{\mathbf{K}}_{\text{Bulk}}^{\text{Ginibre}}$ with a simple form. Let $F_1(z) = 2^{-1}\operatorname{erfc}(z/2)$ and define $\mathbf{K}(z)$ by

$$\mathbf{K}(z) = \begin{pmatrix} -F_1''(z) & -F_1'(z) \\ F_1'(z) & \operatorname{sgn}(z)F_1(|z|) \end{pmatrix}, \tag{2.10}$$

then $\tilde{\mathbf{K}}_{\text{Bulk}}^{\text{Ginibre}}(x,y) = \sqrt{2}\mathbf{K}\left(\sqrt{2}(y-x)\right)$.

Under the edge scaling regime (for definiteness, at the right edge of the spectrum), the limit process of real eigenvalues of the real Ginibre ensemble is a Pfaffian point process on $\mathbb{R}$ with kernel $\mathbf{K}_{\text{Edge}}^{\text{Ginibre}}$ given in [8] by

$$\mathbf{K}_{\text{Edge}}^{\text{Ginibre}}(x,y) = \begin{pmatrix} \partial_1\partial_2 F_2(x,y) & -\partial_1 F_2(x,y) \\ -\partial_2 F_2(x,y) & \frac{1}{2}\operatorname{sgn}(x-y) + F_2(x,y) \end{pmatrix},$$

where

$$F_2(x,y) = \frac{1}{2\pi} \iint_{\mathbb{R}^2 \setminus (-\infty,0)^2} e^{-(x+w_1)^2-(y+w_2)^2} \operatorname{sgn}(w_1 - w_2) \, \mathrm{d}w_1 \mathrm{d}w_2. \tag{2.11}$$

(The original version of [8] has a slightly incorrect derivation of $(\mathbf{K}_{\text{Edge}}^{\text{Ginibre}})_{22}$, see a forthcoming erratum.)

Example 5 (Gaussian ensembles)**.** The most studied of the random matrix en-

sembles are the three classical Gaussian ensembles, which have independent real, complex or quaternion Gaussian entries on and above the diagonal and a suitable symmetry defining the remaining entries. It is convenient to label the ensembles with a parameter $\beta \in \{1, 2, 4\}$ and in each case the ensemble is defined by giving its density with respect to the Lebesgue measure on independent elements. We refer to Deift and Gioev [13] for a full account, which contains the formulae and detailed references.

Gaussian orthogonal ensemble (GOE), $\beta = 1$. The GOE is defined on the space $\mathcal{H}_N$ of real symmetric $N \times N$ matrices $H = (H_{ij})$ by the following probability density

$$\mathrm{d}\mu(H) = c_N^{\mathrm{GOE}} e^{-\operatorname{Tr}(H^2)} \prod_{i=1}^{N} \mathrm{d}H_{ii} \prod_{i<j} \mathrm{d}H_{ij},$$

where c_N^{GOE} is a normalisation constant.

Gaussian unitary ensemble (GUE), $\beta = 2$. The GUE is defined on the space $\mathcal{H}_N$ of complex Hermitian $N \times N$ matrices $H = (H_{ij})$ by the following probability density

$$\mathrm{d}\mu(H) = c_N^{\mathrm{GUE}} e^{-\operatorname{Tr}(H^2)} \prod_{i=1}^{N} \mathrm{d}H_{ii} \prod_{i<j} \mathrm{d}\operatorname{Re}(H_{ij})\, \mathrm{d}\operatorname{Im}(H_{ij}),$$

where c_N^{GUE} is a normalisation constant.

Gaussian symplectic ensemble (GSE), $\beta = 4$. The GSE is defined on the space $\mathcal{H}_{2N}$ of self-dual quaternion $2N \times 2N$ matrices $H = (H_{ij})_{i,j=1}^{N}$, whose 2×2 blocks take the form $H_{ij} = \left(\begin{smallmatrix} \alpha_{ij}+i\beta_{ij} & \gamma_{ij}+i\delta_{jk} \\ -\gamma_{ij}+i\delta_{ij} & \alpha_{ij}-i\beta_{ij} \end{smallmatrix} \right)$ with α_{ij}, β_{ij}, γ_{ij}, δ_{ij} real, by the probability following probability density

$$\mathrm{d}\mu(H) = c_{2N}^{\mathrm{GSE}} e^{-\operatorname{Tr}(H^2)} \prod_{i=1}^{N} \mathrm{d}\alpha_{ii} \prod_{1 \le i < j \le N} \mathrm{d}\alpha_{ij} \mathrm{d}\beta_{ij} \mathrm{d}\gamma_{ij} \mathrm{d}\delta_{ij},$$

where c_{2N}^{GSE} is a normalisation constant.

For each ensemble the matrices are Hermitian, so the N eigenvalues are almost surely real and distinct. Moreover the joint intensities for each eigenvalue process are explicit, having the following density with respect to the Lebesgue measure for

$\beta = 1$, 2 or 4

$$\rho_\beta^{(N)}(x_1, \ldots, x_N) = C_{N,\beta} \prod_{1 \le i < j \le N} |x_j - x_i|^\beta \prod_{i=1}^N e^{-q_\beta x_i^2},$$

where $q_\beta = 1 + 1(\beta = 4)$ and $C_{N,\beta}$ is a normalisation constant. Using orthogonal polynomials, for example, there are kernels for which the intensities can be expressed as determinants or Pfaffians. In particular, the GUE ($\beta = 2$) is a determinantal point process (see example 3) and the GOE/GSE ($\beta = 1$ or 4) are Pfaffian point processes, with explicit kernels dependent on N and β. Under suitable scaling in N, limit processes emerge in the bulk or near the edge of the spectrum as $N \to \infty$ for each β. The appropriate scaling is indicated by Wigner's semicircle law [60], giving that under the normalisation $N^{-1/2}$ the density of eigenvalues in the Gaussian ensembles converges to an explicit semicircle distribution supported on $[-2, 2]$.

Consider scaling limits in the bulk of the spectrum. The sine kernel function is defined by $K^{\text{sine}} : \mathbb{R} \to \mathbb{R}$ by $K^{\text{sine}}(z) = \frac{\sin(\pi z)}{\pi z}$. The scaled GUE eigenvalue process converges to the determinantal point process with kernel $\mathbf{K}^{\text{GUE}}_{\text{Bulk}}(x, y) = K^{\text{sine}}(y - x)$. For scaled GOE/GSE the limiting eigenvalues processes are Pfaffian with kernels

$$\mathbf{K}^{\text{GOE}}_{\text{Bulk}}(x, y) = \begin{pmatrix} -\partial_x K^{\text{sine}}(x - y) & K^{\text{sine}}(x - y) \\ K^{\text{sine}}(x - y) & \int_0^{x-y} K^{\text{sine}}(z)\,\mathrm{d}z - \frac{1}{2}\operatorname{sgn}(x - y) \end{pmatrix},$$
$$\mathbf{K}^{\text{GSE}}_{\text{Bulk}}(x, y) = \begin{pmatrix} -\partial_x K^{\text{sine}}(2(x - y)) & K^{\text{sine}}(2(x - y)) \\ K^{\text{sine}}(2(x - y)) & \int_0^{x-y} K^{\text{sine}}(2z)\,\mathrm{d}z \end{pmatrix}.$$

For scaling limits at the (right) edge of the spectrum, the Airy kernel function $K^{\text{Airy}} : \mathbb{R}^2 \to \mathbb{R}$ is defined in terms of the standard Airy function $\text{Ai} : \mathbb{R} \to \mathbb{R}$ by

$$K^{\text{Airy}}(x, y) = \frac{\text{Ai}(x)\,\text{Ai}'(y) - \text{Ai}'(x)\,\text{Ai}(y)}{x - y},$$
$$\text{Ai}(z) = \frac{1}{\pi} \int_0^\infty \cos\left(\frac{w^3}{3} + wz\right) \mathrm{d}w.$$

The scaled GUE eigenvalue process converges to a determinantal point process with kernel $\mathbf{K}^{\text{GUE}}_{\text{EDGE}}(x, y) = K^{\text{Airy}}(x, y)$. For scaled GOE/GSE the limiting eigenvalue processes are Pfaffian with explicit kernels in terms of K^{Airy} and Ai. See Deift and Gioev [12] for details.

Example 6 (Coalescing and annihilating Brownian motions on $\mathbb{R}$)**.** The primary motivation for this thesis is a model of coalescing and annihilating Brownian mo-

tions. In [59] Tribe and Zaboronski showed that, under a maximal entrance law, instantaneously coalescing Brownian motions and instantaneously annihilating Brownian motions at a fixed time are both Pfaffian point processes on $\mathbb{R}$.

The kernel for annihilating Brownian motions at time $t > 0$ is given by $\mathbf{K}_t^{\mathrm{ABM}}(x, y) = t^{-1/2}\mathbf{K}\left(t^{-1/2}(y - x)\right)$, where $\mathbf{K}(z)$ is defined by (2.10). In particular, as noted in [59], we have the following equivalence of kernels

$$\mathbf{K}_t^{\mathrm{ABM}}(x, y) = \frac{1}{\sqrt{2t}}\tilde{\mathbf{K}}_{\mathrm{Bulk}}^{\mathrm{Ginibre}}\left(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\right). \tag{2.12}$$

In terms of processes, the one-dimensional distribution of annihilating Brownian motions under the maximal entrance law is equivalent to the distribution of real eigenvalues in the bulk limit of the real Ginibre ensemble. By proposition 12 part 1, moving a factor of $t^{-1/2}$ from $(\mathbf{K}_t^{\mathrm{ABM}})_{22}$ to $(\mathbf{K}_t^{\mathrm{ABM}})_{11}$ gives an equivalent kernel $\tilde{\mathbf{K}}_t^{\mathrm{ABM}}$ with

$$\tilde{\mathbf{K}}_t^{\mathrm{ABM}}(x, y) = \begin{pmatrix} -F_t''(y - x) & -F_t'(y - x) \\ F_t'(y - x) & \operatorname{sgn}(y - x)F_t(|y - x|) \end{pmatrix}, \tag{2.13}$$

where $F_t(z) = F_1(z/\sqrt{t}) = 2^{-1}\operatorname{erfc}\left(z/2\sqrt{t}\right)$.

For coalescing Brownian motions the kernels differ by a factor of 2, namely $\mathbf{K}_t^{\mathrm{CBM}}(x, y) = 2\mathbf{K}_t^{\mathrm{ABM}}(x, y)$ and $\tilde{\mathbf{K}}_t^{\mathrm{CBM}}(x, y) = 2\tilde{\mathbf{K}}_t^{\mathrm{ABM}}(x, y)$. The connection between the coalescing and annihilating kernels is a manifestation of the fact that the processes are related via a thinning procedure (see [59] for more details).

2.2.4 Generating new point processes from old

We show that the set of Pfaffian point processes is closed under certain thinning and thickening operations. At the end of the section we present some preliminary remarks on convergence of point processes.

Proposition 15. *Let X be a Pfaffian point process on Λ with a locally bounded kernel $\mathbf{K}$. Fix $\lambda \in [0, 1]$ and let $\tilde{X}$ denote the point process obtained from X by removing each particle independently with probability $1-\lambda$, i.e. the process X thinned at rate λ. Then $\tilde{X}$ is the Pfaffian point process on Λ with kernel $\tilde{\mathbf{K}} = \lambda\mathbf{K}$.*

Proof of proposition 15. For mutually disjoint Borel sets $B_1, \ldots, B_n$, thinning at

rate λ and substituting in the definition of the X intensities $\rho_X^{(n)}$ gives

$$\mathbb{E}\left[\tilde{X}(B_1)\dots\tilde{X}(B_n)\right] = \lambda^n \mathbb{E}\left[X(B_1)\dots X(B_n)\right] = \lambda^n \int_{B_1\times\cdots\times B_n} \rho_X^{(n)}(\mathbf{x})\,\mathrm{d}\nu^n(\mathbf{x}).$$

Note that $\lambda^n \rho_X^{(n)}(\mathbf{x}) = \lambda^n \operatorname{Pf}(\mathbf{K}(\mathbf{x})) = \operatorname{Pf}(\tilde{\mathbf{K}}(\mathbf{x}))$ by proposition 7, so that $\tilde{X}$ is a Pfaffian point process with kernel $\tilde{\mathbf{K}}$. Since $\tilde{\mathbf{K}}$ is locally bounded $\tilde{X}$ is uniquely determined. □

For discrete spaces, we have a simple expression for the intensity and it is straightforward to prove an inhomogeneous extension to the thinning property.

Proposition 16. *Let X be a Pfaffian point process on $\mathbb{Z}$ with a locally bounded kernel $\mathbf{K}$. Fix $\lambda_x \in [0,1]$ for $x \in \mathbb{Z}$ and let $\tilde{X}$ denote the point process obtained from X by removing a particle at x independently with probability $1-\lambda_x$, i.e. the process X thinned at rate λ_x. Then the intensities of $\tilde{X}$ are given, for $\mathbf{x} = (x_1,\dots,x_n)$ with distinct x_i, by*

$$\rho^{\tilde{X}}(\mathbf{x}) = \prod_{i=1}^{n} \lambda_{x_i}\ \rho^X(\mathbf{x}).$$

Equivalently, $\tilde{X}$ is the Pfaffian point process on $\mathbb{Z}$ with kernel $\tilde{\mathbf{K}}(x,y) = \sqrt{\lambda_x\lambda_y}\mathbf{K}(x,y)$.

Proof of proposition 16. The intensity of $\tilde{X}$ is given by

$$\rho^{\tilde{X}}(\mathbf{x}) = \mathbb{E}\left[\tilde{X}(x_1)\dots\tilde{X}(x_n)\right] = \prod_{i=1}^{n}\lambda_{x_i}\,\mathbb{E}\left[X(x_1)\dots X(x_n)\right] = \prod_{i=1}^{n}\lambda_{x_i}\ \rho^X(\mathbf{x}).$$

Substituting in the kernel gives $\rho^{\tilde{X}}(\mathbf{x}) = \prod_{i=1}^n \lambda_{x_i} \operatorname{Pf}(\mathbf{K}(\mathbf{x}))$. The product may be moved inside the Pfaffian by writing it as the determinant of a diagonal matrix B, with entries $B_{2i-1,2i-1} = B_{2i,2i} = \sqrt{\lambda_{x_i}}$, and applying the conjugation formula to $B^T\mathbf{K}(\mathbf{x})B$. We arrive at

$$\rho^{\tilde{X}}(\mathbf{x}) = \operatorname{Pf}\left(\sqrt{\lambda_{x_i}\lambda_{x_j}}\mathbf{K}(x_i,x_j) : i,j \le n\right),$$

whence $\tilde{X}$ is a Pfaffian point process on $\mathbb{Z}$ with kernel $\tilde{\mathbf{K}}(x,y)$. Since the kernel is locally bounded $\tilde{X}$ is uniquely determined. □

The thickening relation is considered separately for $\mathbb{Z}$ and $\mathbb{R}$. The superposition of two point processes is found by combining the particles of each. For simple point processes, to define this unambiguously on a discrete space one must declare what happens at the overlaps, where both processes place particles at the same site. For

$\alpha \in [0,1]$ the *α-superposition* is defined to be the superposition in which each overlap results in a particle with probability α. We define $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.

Proposition 17. *Let X be a Pfaffian point process on $\mathbb{Z}$ with locally bounded kernel $\mathbf{K}(x,y)$ and Y a point process of independent product Bernoulli random variables on $\mathbb{Z}$ with site-dependent rates $\mathbb{P}[Y(x)=1] = 1 - \mathbb{P}[Y(x)=0] = \lambda(x) \in [0,1]$. Fix $\alpha \in [0,1]$ and define $I = \{x \in \mathbb{Z} : 1-(2-\alpha)\lambda(x) = 0\}$. Then for $\mathbf{x} = (x_1, \ldots, x_n)$ with $x_1 < \cdots < x_n$ and $x_i \notin I$ for all $1 \le i \le n$, the α-superposition of X and Y, denoted $(X+Y)_\alpha$, has intensities*

$$\rho^{(X+Y)_\alpha}(\mathbf{x}) = \prod_{i=1}^{n} (1-(2-\alpha)\lambda(x_i)) \operatorname{Pf}\left(\mathbf{K}(\mathbf{x}) + \mathbf{K}_\mu(\mathbf{x})\right), \tag{2.14}$$

where $\mu(x) = \frac{\lambda(x)}{1-(2-\alpha)\lambda(x)}$ and $\mathbf{K}_\mu(x,y) = \mu(x) J 1(x=y)$. More generally, if $x_i \in I$ for some i, then letting $\mathbf{x}_I$ and $\mathbf{x}_{I^c}$ be the vectors formed by the points of $\mathbf{x}$ in and not in I, respectively, we have

$$\rho^{(X+Y)_\alpha}(\mathbf{x}) = \operatorname{Pf}\left(\mathbf{K}_\lambda(\mathbf{x}_I)\right) \prod_{x_i \in I^c} (1-(2-\alpha)\lambda(x_i)) \operatorname{Pf}\left(\mathbf{K}(\mathbf{x}_{I^c}) + \mathbf{K}_\mu(\mathbf{x}_{I^c})\right). \tag{2.15}$$

Equivalently, $(X+Y)_\alpha$ is the Pfaffian point process with kernel

$$\mathbf{K}^{(X+Y)_\alpha}(x,y) = \sqrt{(1-(2-\alpha)\lambda(x))\,(1-(2-\alpha)\lambda(y))}\mathbf{K}(x,y) + \mathbf{K}_\lambda(x,y).$$

For certain choices of parameters the prefactor in $\mathbf{K}^{(X+Y)_\alpha}$ may be complex, so *a priori* this kernel falls outside the current framework. This is not a problem however since the Pfaffian definition immediately extends to complex matrices and the corresponding intensities, which define the point process, are real. This subtlety can be avoided by incorporating the intensity prefactor into the kernel in a different way, but the expression is not as simple.

Proof of proposition 17. Consider first the case that I is empty. The intensity of

$(X+Y)_\alpha$ at $\mathbf{x} = (x_1, \ldots, x_n)$ is

$$\begin{aligned}\rho^{(X+Y)_\alpha}(\mathbf{x}) &= \mathbb{E}\Big[\prod_{i=1}^{n}\Big(X(x_i)(1-\lambda(x_i)) + (1-X(x_i))\,\lambda(x_i) + \alpha X(x_i)\lambda(x_i)\Big)\Big] \\ &= \prod_{i=1}^{n}\big(1-(2-\alpha)\lambda(x_i)\big)\mathbb{E}\Big[\prod_{i=1}^{n}\big(X(x_i)+\mu(x_i)\big)\Big] \\ &= \prod_{i=1}^{n}\big(1-(2-\alpha)\lambda(x_i)\big)\sum_{r=0}^{n}\sum_{|V|=r}\mathbb{E}\Big[\prod_{i\in V}X(x_i)\Big]\prod_{i\in V^c}\mu(x_i),\end{aligned}$$

where the inner sum is over subsets $V \subseteq \{1, 2, \ldots, n\}$ of the particle indices with $|V| = r$. In view of example 1, the V^c product may be written as $\mathrm{Pf}\,(\mathbf{K}_\mu(\mathbf{x})|_U)$, where $U \subseteq \{1, \ldots, 2n\}$ with $|U| = 2(n-r)$ is given by $U = \bigcup_{i\in V^c}\{2i, 2i+1\}$, the subset of matrix indices corresponding to x_i with $i \in V^c$ (recall each x_i has two associated rows and columns). The expectation $\mathbb{E}\Big[\prod_{i\in V}X(x_i)\Big]$ is then given by $\mathrm{Pf}\,(\mathbf{K}(\mathbf{x})|_{U^c})$. Consider $\mathrm{Pf}\,(\mathbf{K}_\mu(\mathbf{x})|_U)$ for a general set $U \subseteq \{1, \ldots, 2n\}$ with $|U|/2 \in \mathbb{N}$. If U is not of the previous form, then there exists a point x_i which only contributes one of its associated rows and columns to $\mathbf{K}_\mu(\mathbf{x})|_U$. Since $\mathbf{K}_\mu(\mathbf{x})$ is block diagonal, this row and column become identically zero in $\mathbf{K}_\mu(\mathbf{x})|_U$, giving $\mathrm{Pf}\,(\mathbf{K}_\mu(\mathbf{x})|_U) = 0$. All together

$$\rho^{(X+Y)_\alpha}(\mathbf{x}) = \prod_{i=1}^{n}\big(1-(2-\alpha)\lambda(x_i)\big)\sum_{U}\mathrm{Pf}\,(\mathbf{K}(\mathbf{x})|_{U^c})\,\mathrm{Pf}\,(\mathbf{K}_\mu(\mathbf{x})|_U)\,,$$

where the sum is over subsets $U \subseteq \{1, \ldots, 2n\}$ with $|U|/2 \in \mathbb{N}$. Expression (2.14) now follows from the Pfaffian summation formula (proposition 4). Indeed the only non-zero terms in the sum are for $U \subseteq \{1, \ldots, 2n\}$ composed of pairs of indices $\{2i-1, 2i\}$, for which the factor $(-1)^{|U|/2}(-1)^{s(U)}$ equals 1.

If I is non-empty, then the intensity is given by

$$\rho^{(X+Y)_\alpha}(\mathbf{x}) = \prod_{x_i\in I}\lambda(x_i)\prod_{x_i\notin I}\big(1-(2-\alpha)\lambda(x_i)\big)\mathbb{E}\Big[\prod_{x_i\notin I}\big(X(x_i)+\mu(x_i)\big)\Big].$$

The product over $x_i \in I$ may be replaced by $\mathrm{Pf}\,(\mathbf{K}_\lambda(\mathbf{x}_I))$ and the terms for $x_i \notin I$ may be treated exactly as above, giving the general expression (2.15).

It remains to obtain the claimed kernel $\mathbf{K}^{(X+Y)_\alpha}$. Suppose that $|\mathbf{x}_{I^c}| = k$. Move the $\mathbf{x}_{I^c}$ prefactor of (2.15) inside the $2k \times 2k$ Pfaffian by applying the conjugation formula (proposition 3) with a $2k \times 2k$ diagonal matrix B, having 2×2 diagonal

blocks $B_{2i-1,2i-1} = B_{2i,2i} = \sqrt{1-(2-\alpha)\lambda(y_i)}$ where $\mathbf{x}_{I^c} = (y_1, \ldots, y_k)$ and $y_1 < \cdots < y_k$. We arrive at

$$\rho^{(X+Y)_\alpha}(\mathbf{x}) = \operatorname{Pf}\big(\mathbf{K}_\lambda(\mathbf{x}_I)\big) \operatorname{Pf}\big(\mathbf{K}^{(X+Y)_\alpha}(\mathbf{x}_{I^c})\big).$$

Noting that $\mathbf{K}^{(X+Y)_\alpha}(x,y) = \mathbf{K}_\lambda(x,y)$ if either of x or y are in I, and $\mathbf{K}_\lambda(x,y) = 0$ unless $x = y$, proposition 10 gives

$$\rho^{(X+Y)_\alpha}(\mathbf{x}) = \operatorname{Pf}\big(\mathbf{K}^{(X+Y)_\alpha}(\mathbf{x})\big),$$

where we recall that the ordering of x_i in the matrix $\mathbf{K}^{(X+Y)_\alpha}(\mathbf{x})$ is irrelevant. This shows that $\mathbf{K}^{(X+Y)_\alpha}(\mathbf{x})$ is a suitable kernel, and since it is locally bounded, the superposition is uniquely determined. □

There is no problem with overlaps in the continuum because we consider independent point processes that do not charge given singletons.

Proposition 18. *Let X be a Pfaffian point process on $\mathbb{R}$ with locally bounded kernel $K(x,y)$ and Y a Poisson point process with intensity measure $\lambda(x)\,\mathrm{d}x$ where $\lambda : \mathbb{R} \to [0,\infty)$ is bounded and $\mathrm{d}x$ is the Lebesgue measure. Then the superposition of X and Y is the Pfaffian point process with kernel*

$$\mathbf{K}(x,y) + \mathbf{K}_\lambda(x,y),$$

where $\mathbf{K}_\lambda(x,y) = \lambda(x) J1(x = y)$.

Proof of proposition 18. Denote the superposition of X and Y by $X + Y$. For mutually disjoint Borel sets $B_1, \ldots, B_n$

$$\mathbb{E}\left[(X+Y)(B_1)\ldots(X+Y)(B_n)\right] = \sum_{r=0}^{n} \sum_{|V|=r} \mathbb{E}\Big[\prod_{i\in V} X(B_i)\Big]\mathbb{E}\Big[\prod_{i\in V^c} Y(B_i)\Big],$$

where the inner sum is over subsets $V \subseteq \{1, 2 \ldots, n\}$ of the particle indices with $|V| = r$. In view of example 2, the intensity for Y at $\mathbf{x} = (x_1, \ldots, x_n)$ is $\prod_{i=1}^n \lambda(x_i)$, which may be written as the Pfaffian of $\mathbf{K}_\lambda(\mathbf{x})$. Exactly as for the discrete analogue, $\operatorname{Pf}\left(\mathbf{K}_\lambda(\mathbf{x})|_U\right)$ vanishes unless the subset of matrix indices $U \subseteq \{1, \ldots, 2n\}$ is composed of pairs $\{2i-1, 2i\}$. Substituting in the intensities, the previous display is equal to

$$\int_{B_1\times\cdots\times B_n} \sum_U \operatorname{Pf}(\mathbf{K}(\mathbf{x})|_{U^c}) \operatorname{Pf}\left(\mathbf{K}_\lambda(\mathbf{x})|_U\right) \mathrm{d}x_1 \ldots \mathrm{d}x_n,$$

where the sum is over subsets $U \subseteq \{1, \ldots, 2n\}$ with $|U|/2 \in \mathbb{N}$. The integrand gives the intensity for $X + Y$, which by the summation formula (proposition 4) may be rewritten as the single Pfaffian $\mathrm{Pf}\,(\mathbf{K}(\mathbf{x}) + \mathbf{K}_\lambda(\mathbf{x}))$, giving the claimed kernel. Since $\mathbf{K}(x, y) + \mathbf{K}_\lambda(x, y)$ is locally bounded, the superposition is uniquely determined. □

Finally, we comment on the important operation that is convergence of Pfaffian point processes. This tool is brought to the fore in chapter 4, developing continuum point processes as limits of discrete approximations. We delay results until then, with the remainder of this section dedicated to reviewing a suitable notion of convergence for point processes. Firstly, all of the point processes we consider live on subsets of $\mathbb{R}$, namely $\epsilon\mathbb{Z}$ or $\mathbb{R}$ itself, so can be viewed as random measures on $\mathcal{M}_0(\mathbb{R}) \subset \mathcal{M}_{\mathrm{LFP}}(\mathbb{R})$. An appropriate notion of convergence for random measures is in distribution, that is, convergence of the expectations of certain functionals. A sequence of random measures $X^{(n)}$ converges to X in distribution, on $\mathcal{M}_{\mathrm{LFP}}(\mathbb{R})$ equipped with the topology of vague convergence, if

$$\mathbb{E}[f(X^{(n)})] \to \mathbb{E}\,[f(X)]\,,$$

for all bounded continuous $f : \mathcal{M}_{\mathrm{LFP}}(\mathbb{R}) \to \mathbb{R}$. The space $\mathcal{M}_{\mathrm{LFP}}(\mathbb{R})$ is Polish in the vague topology. (Indeed the set of locally finite measures $\mathcal{M}_{\mathrm{LF}}(\mathbb{R})$ is Polish and $\mathcal{M}_{\mathrm{LFP}}(\mathbb{R})$ is a closed subset.) However the set of simple measures $\mathcal{M}_0(\mathbb{R})$ is not closed in $\mathcal{M}_{\mathrm{LFP}}(\mathbb{R})$ and one must check that the limit of a convergent sequence on $\mathcal{M}_0(\mathbb{R})$ remains supported on $\mathcal{M}_0(\mathbb{R})$. Note that this formulation of convergence is not particularly practical, since we do not have a deep intuition for functionals defined on $\mathcal{M}_{\mathrm{LFP}}(\mathbb{R})$. We are more familiar with convergence in distribution on $\mathbb{R}$. Fortunately, there is an equivalent formulation requiring convergence in distribution of the random variables

$$X^{(n)} f \to X f, \qquad \text{for } f \in C_0(\mathbb{R}).$$

Expanding, a sequence $X^{(n)}$ converges to X in distribution, on $\mathcal{M}_{\mathrm{LFP}}(\mathbb{R})$ equipped with the topology of vague convergence, if

$$\mathbb{E}[h(X^{(n)} f)] \to \mathbb{E}\,[h(X f)]\,,$$

for all bounded continuous $h : \mathbb{R} \to \mathbb{R}$ and $f \in C_0(\mathbb{R})$. Since our point processes have unbounded support, we require functions f with compact support to ensure the integrals are finite. A stronger notion, weak convergence, corresponds to bounded

and continuous test functions. See Kallenberg [32], for example, for a thorough exposition.

Chapter 3

Discrete models

We begin our study of one-dimensional interacting particle systems with models defined on $\mathbb{Z}$. The discrete particle-space facilitates a convenient framework for rigorous construction and analysis of processes. Whilst interesting in their own right, these models also form the basis for investigating continuum analogues by taking scaling limits of discrete approximations, see chapter 4. It is satisfactory that Pfaffian structure already appears at the discrete level.

Section 3.1 provides a whistle-stop tour of discrete-space interacting particle systems, as well as contextualising the essentials of Pfaffian point processes and introducing notation. The core models of coalescing and annihilating random walks are the subject of section 3.2. In section 3.2.1 the models are defined, along with the statement of the main result identifying them as Pfaffian point processes. The proof of this result is given in section 3.2.2. In section 3.3 two extensions to the pure interaction models are shown to be Pfaffian. The first, in section 3.3.1, is annihilating random walks with pairwise immigration. The proof of the Pfaffian result is a small extension to that for the core model of pure annihilation. The second extension, in section 3.3.2, is coalescing random walks with branching. This case is more subtle, with delicate conditions on the rates, and the proof for the core model of pure coalescence requires a non-trivial modification. In section 3.3.3 the two extended models are shown to be related, for particular initial conditions, via a generalised thinning relation that also involves thickening.

3.1 Interacting particle systems on $\mathbb{Z}$

The backbone of the thesis is interacting particle systems on $\mathbb{Z}$. Intuitively by this we simply mean a collection of particles at sites of $\mathbb{Z}$ whose evolutions in time

may depend on other particles. In the spirit and notation of Liggett [39] a natural way to describe the system is through *occupation variables* $\eta_t(x)$, where $\eta_t(x) = 1$ indicates that site $x \in \mathbb{Z}$ is occupied at time t and $\eta_t(x) = 0$ for unoccupied. The collection $\eta_t = (\eta_t(x) : x \in \mathbb{Z}) \in \{0,1\}^{\mathbb{Z}}$ is the whole particle configuration at time t. As interactions occur η_t evolves in $\{0,1\}^{\mathbb{Z}}$. We are only concerned with time-homogeneous Markovian dynamics and we therefore consider an interacting particle system on $\mathbb{Z}$ as a time-homogeneous Markov process $(\eta_t : t \geq 0)$ with values in $\{0,1\}^{\mathbb{Z}}$.

We briefly review some classical theory, referring to Liggett [39] and Rogers and Williams [45] for detailed accounts. Let $(\eta_t : t \geq 0)$ be a Markov process on $\{0,1\}^{\mathbb{Z}}$, equipped with the discrete topology and Borel σ-algebra. The process may be described by its transition operators on functionals. To wit, a semigroup of operators $(P_t : t \geq 0)$ acting on bounded measurable functions $F : \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$ is defined by $P_t F(\eta) = \mathbb{E}\left[F(\eta_t) | \eta_0 = \eta\right]$ for $\eta \in \{0,1\}^{\mathbb{Z}}$. Indicating the initial condition in the subscript, the expectation is also written $P_t F(\eta) = \mathbb{E}_\eta\left[F(\eta_t)\right]$. Such transition operators are rarely explicit, but we can also consider infinitesimal movements. A key concept is the *infinitesimal generator* LF defined by $LF(\eta) = \lim_{t \downarrow 0}(P_t F(\eta) - F(\eta))/t$ whenever the limit exists. The set of functions for which the limit exists is called the *domain of the generator*. The generator encodes much information on the Markov process $(\eta_t : t \geq 0)$, for example the connection to derivatives

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}_{\eta_0}\left[F(\eta_t)\right] = L\mathbb{E}_{\eta_0}\left[F(\eta_t)\right] = \mathbb{E}_{\eta_0}\left[LF(\eta_t)\right], \tag{3.1}$$

where the operator in central expression acts on the expectation as a function of the initial condition. The first equality is the Kolmogorov backward equation, and the second equality is due to the Markov semigroup and its generator commuting. On a finite state space the generator determines the process and takes the form

$$LF(\eta) = \sum_{\eta'} \omega(\eta, \eta')(F(\eta') - F(\eta)), \tag{3.2}$$

where $\omega(\eta, \eta')$ is the rate at which the process jumps from η to η'. For an uncountable state space we are outside the standard theory and it is not *a priori* clear whether the whole process is determined uniquely by the local dynamics. For general state spaces, Liggett [39] gives conditions on the rates that ensure (3.2) holds at least for $F : \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$ depending on only finitely many coordinates, so that the generator determines a unique Markov process. In other words, under the rate conditions

the formula for L is defined on a big enough class of functions F to determine a semigroup, then there exists a unique Markov process with this semigroup (see [45] section III, for example).

The processes we consider have nearest neighbour interactions. The conditions of Liggett ([39], p. 27) demand uniform control of the total transition rate for subsets involving a given site and uniform control of the smoothness of transition rates as functions of configurations. Both are simple to verify in the case of nearest neighbour interactions when all rates $\omega(\eta, \eta')$ are uniformly bounded.

To summarise, we may define the law of a unique time-homogeneous Markov process on $\mathbb{Z}$, hence an interacting particle system, by a generator of the form (3.2) provided the (nearest neighbour interaction) rates are uniformly bounded. In particular (3.1) and (3.2) hold for functions depending on finitely many coordinates.

We finish by connecting interacting particle systems to Pfaffian point processes. With (2.1) in mind, a variable η with values in $\{0,1\}^{\mathbb{Z}}$ can be interpreted as a simple point process on $\mathbb{Z}$. Expression (2.2) for the intensities in discrete space leads to the following definition of a Pfaffian point process in this context: there exists a kernel $\mathbf{K} : \mathbb{Z}^2 \to \mathbb{R}^{2\times 2}$ such that for distinct $x_1, \ldots, x_n \in \mathbb{Z}$

$$\mathbb{E}\left[\eta(x_1)\ldots\eta(x_n)\right] = \mathrm{Pf}(\mathbf{K}(x_i, x_j) : i, j \leq n).$$

The kernel $\mathbf{K}$ has the form $\mathbf{K}(x,y) = \begin{pmatrix} \mathbf{K}_{11}(x,y) & \mathbf{K}_{12}(x,y) \\ \mathbf{K}_{21}(x,y) & \mathbf{K}_{22}(x,y) \end{pmatrix}$ for some $\mathbf{K}_{ij} : \mathbb{Z}^2 \to \mathbb{R}$ satisfying $\mathbf{K}_{ij}(x,y) = -\mathbf{K}_{ji}(y,x)$ for all $i, j \in \{1,2\}$ and $x, y \in \mathbb{Z}$.

Finally, for an interacting particle system $(\eta_t : t \geq 0)$ we can ask whether at each fixed time the point process η_t is Pfaffian. In the current framework a single-time description is the best we can hope for. We remark here that there is a multi-time extension of the theory of Pfaffian point processes to deal with a full stochastic process $(\eta_t : t \geq 0)$, involving multi-time intensities and multi-time kernels. We return to this concept in Chapter 5.

3.2 Coalescing and annihilating random walks

A class of interacting particle systems on $\mathbb{Z}$, involving instantaneously annihilating or coalescing random walks, are shown to be Pfaffian point processes for a large set of initial conditions, including deterministic.

In [59], systems of instantly coalescing, or instantly annihilating, Brownian motions, under a maximal entrance law, were shown to be Pfaffian point processes at any fixed time $t > 0$ (see example 6). The aim of this section is to generalize this result in the following ways: we consider (i) analogous particle systems on $\mathbb{Z}$; (ii) spatially inhomogeneous nearest neighbour motion; (iii) general deterministic initial conditions. The Pfaffian point process structure survives all of these changes.

3.2.1 Definition of models and statement of results

The models involve coalescence and annihilation of particles. In a coalescent system there would be an instantaneous coalescence where the two particles merge to leave a single particle; in an annihilating system there would be an instantaneous annihilation where both particles disappear. We consider a mixed system, whose dynamics we now describe informally before explicitly defining the generator.

Between interactions all particles jump independently following a nearest neighbour random walk on $\mathbb{Z}$, jumping

$$x \to x-1 \text{ at rate } q_x, \qquad \text{and} \qquad x-1 \to x \text{ at rate } p_x.$$

The parameter $\theta \in [0,1]$ is fixed, and when two particles interact they instantaneously annihilate with probability θ or coalesce with probability $1-\theta$.

These Markovian dynamics are encoded carefully in the generator of the process. For suitable $F : \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$, the generator is given by

$$\begin{aligned}\mathcal{L}F(\eta) = \sum_{x\in\mathbb{Z}} q_x \left(\theta F(\eta^a_{x,x-1}) + (1-\theta)F(\eta^c_{x,x-1}) - F(\eta)\right) \\ + \sum_{x\in\mathbb{Z}} p_x \left(\theta F(\eta^a_{x-1,x}) + (1-\theta)F(\eta^c_{x-1,x}) - F(\eta)\right),\end{aligned}$$

where $\eta^a_{x,y}$ (respectively $\eta^c_{x,y}$) is the new configuration after a jump from site x to y followed by instantaneous annihilation (respectively coalescence). These are defined, when $x \neq y$, by

$$\begin{cases} \eta^a_{x,y}(z) = \eta^c_{x,y}(z) = \eta(z) & \text{for } z \notin \{x,y\}, \\ \eta^a_{x,y}(x) = \eta^c_{x,y}(x) = 0, \\ \eta^a_{x,y}(y) = (\eta(x)+\eta(y)) \bmod(2), \\ \eta^c_{x,y}(y) = \min\{1, \eta(x)+\eta(y)\}. \end{cases}$$

We take $(p_x, q_x \geq 0 : x \in \mathbb{Z})$ uniformly bounded, then this generator determines the law of a unique Markov process, for any given initial condition $\eta_0 \in \{0,1\}^{\mathbb{Z}}$, which we refer to as *coalescing and annihilating random walks (CARW)*. We denote its law by $\mathbb{P}_{\eta_0}$ and $\mathbb{E}_{\eta_0}$ on path space with canonical variables $(\eta_t : t \geq 0)$.

By choosing $\theta = 0$ or $\theta = 1$, our results apply to both purely coalescing and purely annihilating systems, referred to as *coalescing random walks (CRW)* and *annihilating random walks (ARW)*, respectively. We note that the mixed system arises as the dual process to the multi-valued voter model (also known as the Potts model) started from Bernoulli initial conditions, where there are $q \geq 2$ colours and $\theta = 1/(q-1)$ (see [14], [15]). The spatially inhomogeneous version of the process occurs in studies on reaction diffusion models with quenched disorder (see [37]).

The matrix kernel for CARW is constructed from a single scalar function $(K_t(x,y) : x \leq y)$ defined as follows. For $x, y \in \mathbb{Z}$ with $x \leq y$, and for $\eta \in \{0,1\}^{\mathbb{Z}}$, we define

$$\eta[x,y) = \sum_{x \leq z < y} \eta(z) \quad \text{if } x < y, \tag{3.3}$$

and $\eta[x,x) = 0$, and we define the 'spin pair' by

$$\sigma_{x,y}(\eta) = (-\theta)^{\eta[x,y)}.$$

We use the convention that $0^0 = 1$ so that when $\theta = 0$ the spin pair reduces to the indicator of an empty interval, that is $\sigma_{x,y}(\eta) = 1(\eta[x,y) = 0)$. We now set

$$K_t(x,y) = \mathbb{E}_{\eta_0}\left[\sigma_{x,y}(\eta_t)\right], \qquad \text{for } t \geq 0,\ x, y \in \mathbb{Z} \text{ with } x \leq y. \tag{3.4}$$

We also need the difference operators D^+ and D^-, defined for $f : \mathbb{Z} \to \mathbb{R}$ by

$$D^+ f(x) = f(x+1) - f(x), \qquad D^- f(x) = f(x-1) - f(x). \tag{3.5}$$

The notation $D_i^{\pm}$ means that the operator $D^{\pm}$ is applied in the i-th variable. We can now state the main result for CARW.

Theorem 1. *For any initial condition $\eta_0 \in \{0,1\}^{\mathbb{Z}}$, and at any fixed time $t \geq 0$, the CARW variable η_t is a Pfaffian point process on $\mathbb{Z}$ with kernel $\mathbf{K}$ given, for $x < y$, by*

$$\mathbf{K}(x,y) = \frac{-1}{1+\theta} \begin{pmatrix} K_t(x,y) & D_2^+ K_t(x,y) \\ D_1^+ K_t(x,y) & D_1^+ D_2^+ K_t(x,y) \end{pmatrix} \tag{3.6}$$

and $\mathbf{K}_{12}(x,x) = \frac{-1}{1+\theta} D_2^+ K_t(x,x)$*, and other entries determined by the symmetry conditions.*

Although the theorem is stated for deterministic initial conditions, under certain random choices, including when all sites $(\eta_0(x) : x \in \mathbb{Z})$ are independent, the variable η_t remains a Pfaffian point process (see remark 6 after the proof of theorem 1).

The scalar function $K_t(x,y)$ that underlies the Pfaffian matrix kernel $\mathbf{K}$ can be characterized as the solution to a system of differential equations indexed over part of the lattice. We define a one-particle generator L, acting on $f : \mathbb{Z} \to \mathbb{R}$, by

$$Lf(x) = q_x D^+ f(x) + p_x D^- f(x). \tag{3.7}$$

We will show that the function $(K_t(x,y) : t \geq 0,\, x,y \in \mathbb{Z},\, x < y)$ is the unique bounded solution to the equation

$$\begin{cases} \partial_t K_t(x,y) &= (L_x + L_y) K_t(x,y) & \text{for } x < y,\, t > 0, \\ K_t(x,x) &= 1 & \text{for all } x,\, t > 0, \\ K_0(x,y) &= \sigma_{x,y}(\eta_0) & \text{for } x \leq y. \end{cases} \tag{3.8}$$

The notation L_x is used to indicate that the operator L acts on the x variable.

Remark 3. Instantly coalescent systems and instantly annihilating systems are related by a well known thinning relation (see corollary 1, also [59] section 2.1 or [6]). This can be extended to show that CRW can be thinned to give the CARW system (and as expected one thins by a factor $1/(1+\theta)$). Thinning also acts naturally on Pfaffian point processes (see proposition 15), changing the underlying kernel by the same factor. However, this connection seems to relate the two systems only when the initial conditions are similarly related by thinning, and so does not apply to a deterministic initial condition for our process.

3.2.2 Proof of theorem 1

Sketch of argument. The proof of the theorem follows similar lines to [59]. The key tool is a Markov duality. Indeed for any $n \geq 1$ the product of n spin pairs $\eta \mapsto \sigma_{x_1,x_2}(\eta) \ldots \sigma_{x_{2n-1},x_{2n}}(\eta)$ is a suitable Markov duality function, as shown in Lemma 1. Exploiting this allows us to calculate the expectations

$$\mathbb{E}_{\eta_0}\left[\sigma_{x_1,x_2}(\eta_t) \ldots \sigma_{x_{2n-1},x_{2n}}(\eta_t)\right],$$

as the solutions of $2n$-dimensional (spatially inhomogeneous) lattice heat equations. This is similar to the Markov dualities used in [7] to study the ASEP and q-TASEP models.

The dual process can be taken to be (a spatially inhomogeneous version of) the one-dimensional Glauber spin chain (see remark 4 after lemma 1). This model is known to be solvable by mapping to a system of free fermions operators (see [20, 21]). Fermions are naturally associated to Pfaffians, and it turns out that the duality expectations are given by $2n \times 2n$ Pfaffians of a matrix built from a scalar kernel $K_t(x_i, x_j)$, as shown in lemma 2.

The final step is to reconstruct the particle intensities from the product spin expectations. This is possible via the identity

$$\eta(x) = \frac{1 - \sigma_{x,x+1}(\eta)}{1 + \theta}. \tag{3.9}$$

This leads to a linear reconstruction formula for the n-point intensity in terms of the $2n \times 2n$ Pfaffians for the product of n spin pairs. The Pfaffian structure is preserved by the reconstruction formula, with the matrix breaking into 2×2 blocks corresponding to the spin pairs, and this yields the desired matrix kernel $\mathbf{K}(x_i, x_j)$.

Proof of theorem 1. The key to the argument is the following Markov duality function. For $n \geq 1$ and $\mathbf{x} = (x_1, \ldots, x_{2n})$ with $x_1 \leq x_2 \leq \cdots \leq x_{2n}$ we define the *product spin function* by

$$\Sigma_{\mathbf{x}}(\eta) = \prod_{i=1}^{n} \sigma_{x_{2i-1}, x_{2i}}(\eta).$$

Note that $\Sigma_{\mathbf{x}}(\eta)$ depends only on finitely many coordinates of η and so lies in the domain of the generator $\mathcal{L}$. The Markov duality is encoded in the following generator calculation.

Lemma 1. *For $\mathbf{x} = (x_1, \ldots, x_{2n})$ with $x_1 < x_2 < \cdots < x_{2n}$ the action of the particle generator $\mathcal{L}$ on $\Sigma_{\mathbf{x}}(\eta)$ is*

$$\mathcal{L}\Sigma_{\mathbf{x}}(\eta) = \sum_{i=1}^{2n} L_{x_i} \Sigma_{\mathbf{x}}(\eta),$$

where L_{x_i}, given by (3.7), acts on the coordinate x_i in $\Sigma_{\mathbf{x}}$.

Remark 4. We do not make use of a dual Markov process, but this lemma could be cast into the standard framework (see Ethier and Kurtz [17] chapter 4) relating

two Markov processes. The dual process can be taken to be a finite system of particles with motion generator L that are instantly annihilating (with state space the disjoint union $\bigcup_{m=0}^{n} \mathbb{R}^{2m}$). This annihilating system describes the motion of domain walls in the Ising spin chain with (a spatially inhomogeneous version of) the Glauber dynamics [27] and the dual process could also be taken to be this spin chain. The formulae connecting a set of spins ($\sigma(x) \in \{-1, +1\}, x \in \mathbb{Z}$) to the occupation variables ($\eta(x) \in \{0, 1\}, x \in \mathbb{Z}$) of domain walls, where $\eta(x) = 1$ indicates different spins at x and $x + 1$, are

$$\eta(x) = \frac{1 - \sigma(x)\sigma(x+1)}{2}, \qquad (-1)^{\eta[x,y)} = \sigma(x)\sigma(y).$$

We do not exploit the link between the spin chain and annihilating systems but it is the origin of our use of the term 'spin pair'.

Proof of lemma 1. Each term in the generator $\mathcal{L}$ involves a modified configuration, $\eta^a_{x,y}$ or $\eta^c_{x,y}$, which differs from η on at most two neighbouring sites. The condition that $x_i < x_{i+1}$ ensures that this modified configuration will agree with η on all but at most one of the intervals $[x_{2i-1}, x_{2i})$, and hence the value of at most one of the spins $\sigma_{x_{2i-1},x_{2i}}$ will change. This allows us to separate the action of the generator as follows

$$\mathcal{L}\,\Sigma_{\mathbf{x}}(\eta) = \sum_{i=1}^{n} \left(\prod_{j=1, j\neq i}^{n} \sigma_{x_{2j-1},x_{2j}}(\eta) \right) \mathcal{L}\,\sigma_{x_{2i-1},x_{2i}}(\eta). \tag{3.10}$$

We turn our attention to a single spin $\sigma_{x,y}(\eta)$. Fix $x < y$ and consider the part of the generator

$$\sum_{z\in\mathbb{Z}} q_z \left(\theta\sigma_{x,y}(\eta^a_{z,z-1}) + (1-\theta)\sigma_{x,y}(\eta^c_{z,z-1}) - \sigma_{x,y}(\eta) \right),$$

corresponding to left jumps. The terms in this sum indexed by $z \leq x - 1$ and by $z \geq y + 1$ are zero, as the modified configurations are unchanged in the interval $[x, y)$. The terms corresponding to $x + 1 \leq z \leq y - 1$ are also zero since we claim that

$$\theta\sigma_{x,y}(\eta^a_{z,z-1}) + (1-\theta)\sigma_{x,y}(\eta^c_{z,z-1}) - \sigma_{x,y}(\eta) = 0.$$

Indeed, since $\{z - 1, z\} \in [x, y)$, the left hand side is proportional to

$$\theta(-\theta)^{\eta(z-1)+\eta(z) \bmod(2)} + (1-\theta)(-\theta)^{\min(1,\eta(z-1)+\eta(z))} - (-\theta)^{\eta(z-1)+\eta(z)},$$

and checking the three cases $\eta(z-1)+\eta(z) \in \{0, 1, 2\}$ shows that this is always zero.

(This identity is similar to a key quadratic identity behind the ASEP dualities in [7].) Thus jumps between sites both lying outside or both lying inside the interval $[x, y)$ give zero contribution to the generator and only the terms when $z = x$ or $z = y$, where a jump crosses an endpoint of the interval, contribute. It was this key property that was looked for when trying to find a duality function.

For the two remaining terms, when $z = x$ or $z = y$, two similar short calculations lead to

$$\theta\sigma_{x,y}(\eta^a_{x,x-1}) + (1-\theta)\sigma_{x,y}(\eta^c_{x,x-1}) - \sigma_{x,y}(\eta) = \sigma_{x+1,y}(\eta) - \sigma_{x,y}(\eta) = D^+_x\sigma_{x,y}(\eta),$$

$$\theta\sigma_{x,y}(\eta^a_{y,y-1}) + (1-\theta)\sigma_{x,y}(\eta^c_{y,y-1}) - \sigma_{x,y}(\eta) = \sigma_{x,y+1}(\eta) - \sigma_{x,y}(\eta) = D^+_y\sigma_{x,y}(\eta).$$

Collecting the terms from all possible $z \in \mathbb{Z}$ gives

$$\sum_{z\in\mathbb{Z}} q_z \left(\theta\sigma_{x,y}(\eta^a_{z-1,z}) + (1-\theta)\sigma_{x,y}(\eta^c_{z-1,z}) - \sigma_{x,y}(\eta)\right) \\ = q_x D^+_x\sigma_{x,y}(\eta) + q_y D^+_y\sigma_{x,y}(\eta).$$

Similar calculations for the terms corresponding to right jumps show that

$$\sum_{z\in\mathbb{Z}} p_z \left(\theta\sigma_{x,y}(\eta^a_{z-1,z}) + (1-\theta)\sigma_{x,y}(\eta^c_{z-1,z}) - \sigma_{x,y}(\eta)\right) \\ = p_x D^-_x\sigma_{x,y}(\eta) + p_y D^-_y\sigma_{x,y}(\eta),$$

and hence

$$\mathcal{L}\,\sigma_{x,y}(\eta) = (L_x + L_y)\,\sigma_{x,y}(\eta).$$

Using this in (3.10) completes the proof. □

For $\mathbf{x} = (x_1, \ldots, x_{2n})$ with $x_1 < \cdots < x_{2n}$, the expectation $u(t, \mathbf{x}, \eta_0) = \mathbb{E}_{\eta_0}[\Sigma_{\mathbf{x}}(\eta_t)]$ satisfies

$$\partial_t u(t, \mathbf{x}, \eta_0) = \mathbb{E}_{\eta_0}[\mathcal{L}\Sigma_{\mathbf{x}}(\eta_t)] = \mathbb{E}_{\eta_0}\left[\sum_{i=1}^{2n} L_{x_i}\Sigma_{\mathbf{x}}(\eta_t)\right] = \sum_{i=1}^{2n} L_{x_i}u(t, \mathbf{x}, \eta_0). \quad (3.11)$$

Indeed the first equality comes from the Kolmogorov backward equation (3.1) for the Markov process $(\eta_t : t \geq 0)$, the second equality is due to lemma 1, and the final equality is interchanging a finite sum and expectation. Thus the duality allows us to recast an infinite dimensional Kolmogorov equation in (t, η_0) as a finite dimensional

ODE in $(t, \mathbf{x})$. We now show that this ODE is exactly solved by a Pfaffian.

Lemma 2. *For all $\eta_0 \in \{0,1\}^{\mathbb{Z}}$, for all $n \geq 1$, $\mathbf{x} = (x_1, \ldots, x_{2n})$ with $x_1 \leq \cdots \leq x_{2n}$ and $t \geq 0$,*

$$\mathbb{E}_{\eta_0}\left[\Sigma_{\mathbf{x}}(\eta_t)\right] = \mathrm{Pf}(K^{(2n)}(t, \mathbf{x})),$$

where $K^{(2n)}(t, \mathbf{x})$ is the anti-symmetric $2n \times 2n$ matrix with entries $K_t(x_i, x_j)$ for $i < j$ for the function K_t defined in (3.4), that is $K_t(x, y) = \mathbb{E}_{\eta_0}[\sigma_{x,y}(\eta_t)]$ for $x \leq y$.

Proof of lemma 2. For $n \geq 1$ denote

$$\begin{aligned}
V_{2n} &= \{\mathbf{x} \in \mathbb{Z}^{2n} : x_1 < \cdots < x_{2n}\},\\
\overline{V}_{2n} &= \{\mathbf{x} \in \mathbb{Z}^{2n} : x_1 \leq \cdots \leq x_{2n}\},\\
\partial V_{2n}^{(i)} &= \{\mathbf{x} \in \mathbb{Z}^{2n} : x_1 < \cdots < x_i = x_{i+1} < x_{i+2} < \ldots x_{2n}\},\\
\partial V_{2n} &= \bigcup_{i=1}^{2n-1} \partial V_{2n}^{(i)}.
\end{aligned}$$

We now detail a system of ODEs indexed by V_{2n}, which will involve driving terms indexed by ∂V_{2n}. Fix an initial condition $\eta_0 \in \{0,1\}^{\mathbb{Z}}$ and $n \geq 1$, then define

$$u^{(2n)}(t, \mathbf{x}) = \mathbb{E}_{\eta_0}\left[\Sigma_{\mathbf{x}}(\eta_t)\right] \qquad \text{for } t \geq 0, \text{ and } \mathbf{x} \in \overline{V}_{2n}.$$

For $n \geq 1$, $u^{(2n)}$ solves the following system of ODEs:

$$(\mathrm{ODE})_{2n} \qquad \begin{cases} \partial_t u^{(2n)}(t, \mathbf{x}) &= \sum_{i=1}^{2n} L_{x_i} u^{(2n)}(t, \mathbf{x}) & \text{on } [0, \infty) \times V_{2n},\\ u^{(2n)}(t, \mathbf{x}) &= u^{(2n-2)}(t, \mathbf{x}^{i,i+1}) & \text{on } [0, \infty) \times \partial V_{2n}^{(i)},\\ u^{(2n)}(0, \mathbf{x}) &= \Sigma_{\mathbf{x}}(\eta_0) & \text{on } V_{2n}. \end{cases}$$

The notation $\mathbf{x}^{i,i+1}$ is for the vector $\mathbf{x}$ with coordinates x_i and x_{i+1} removed. Thus, when $n \geq 2$, for $\mathbf{x} \in \partial V_{2n}^{(i)}$ we have $\mathbf{x}^{i,i+1} \in V_{2n-2}$. $(\mathrm{ODE})_{2n}$ is a system of ODEs indexed over V_{2n}. For $n \geq 2$, to evaluate $L_{x_i} u^{(2n)}$ one may need the values of $u^{(2n)}$ at some points $\mathbf{x} \in \partial V_{2n}^{(i)}$, which then act as driving functions for the differential equation. The second equation, which we call the boundary condition, states that these can be deduced from the values of $u^{(2n-2)}$. Indeed the boundary condition follows simply from the fact that

$$\Sigma_{\mathbf{x}}(\eta) = \Sigma_{\mathbf{x}^{i,i+1}}(\eta) \qquad \text{for } \mathbf{x} \in \partial V_{2n}^{(i)} \text{ and } n \geq 2.$$

By setting $u^{(0)} = 1$, the equation also holds for $n = 1$, encoding the fact that $u^{(2)}(t, (x, x)) = 1$ for all $t \geq 0$, $x \in \mathbb{Z}$.

The infinite sequence of equations $((\mathrm{ODE})_{2n} : n = 1, 2, \dots)$ are uniquely solvable, within the class of continuously differentiable functions satisfying

$$\sup_{t\geq 0} \sup_{\mathbf{x}\in V_{2n}} |u^{(2n)}(t, \mathbf{x})| < \infty.$$

Indeed the boundary condition for $u^{(2)}$ is simply that $u^{(2)}(t, (x, x)) = 1$, and standard (weighted) Gronwall estimates show uniqueness of solutions of $(\mathrm{ODE})_2$. Inductively, the boundary condition for $u^{(2n)}$ is given by the uniquely determined values of $u^{(2n-2)}$ and hence $u^{(2n)}$ can be found uniquely from $(\mathrm{ODE})_{2n}$. See appendix A for details.

We now check that $(\mathrm{Pf}(K^{(2n)}(t, \mathbf{x})) : n = 1, 2, \dots)$ also satisfies $((\mathrm{ODE})_{2n} : n = 1, 2, \dots)$. First we consider the initial conditions. Fix $\mathbf{x} \in V_{2n}$ and choose $x_0 \leq x_1$. For $\theta > 0$ the entries in the Pfaffian at time zero can be rewritten as quotients, namely for $i < j$

$$K_0(x_i, x_j) = \sigma_{x_i,x_j}(\eta_0) = \frac{(-\theta)^{\eta_0[x_0,x_j)}}{(-\theta)^{\eta_0[x_0,x_i)}}.$$

Proposition 9 shows that

$$\mathrm{Pf}(K^{(2n)}(0, \mathbf{x})) = \prod_{i=1}^{n} \frac{(-\theta)^{\eta_0[x_0,x_{2i-1})}}{(-\theta)^{\eta_0[x_0,x_{2i})}} = \Sigma_{\mathbf{x}}(\eta_0), \tag{3.12}$$

and by letting $\theta \downarrow 0$ the identity is true when $\theta = 0$.

Next we check the boundary conditions. We fix $\mathbf{x} \in \partial V_{2n}^{(i)}$, $t \geq 0$ and write $K^{(2n)}$ for the matrix $K^{(2n)}(t, \mathbf{x})$. By conjugating with a suitable elementary matrix B, the matrix

$$\hat{K}^{(2n)} = B^T K^{(2n)} B$$

is the result of subtracting row $i+1$ from row i, and column $i+1$ from column i. The Pfaffian conjugation identity (proposition 3) ensures that $\mathrm{Pf}(\hat{K}^{(2n)}) = \mathrm{Pf}(K^{(2n)})$. However the equality $x_i = x_{i+1}$ implies that the i-th row of $\hat{K}^{(2n)}$ has all zero entries except for $\hat{K}^{(2n)}_{i\,i+1} = 1$. Expanding the Pfaffian of $\hat{K}^{(2n)}$ along row i (using proposition 5) shows that, when $n \geq 2$,

$$\mathrm{Pf}(\hat{K}^{(2n)}(t, \mathbf{x})) = \mathrm{Pf}(K^{(2n-2)}(t, \mathbf{x}^{i,i+1})).$$

When $n = 1$ we obtain $\mathrm{Pf}(\hat{K}^{(2)}) = 1$. This is exactly the desired boundary condition.

Finally we check the differential equation in $(\text{ODE})_{2n}$. Note that the entries in $K^{(2n)}(t,\mathbf{x})$ solve $(\text{ODE})_2$, that is $\partial_t K_t(x_i,x_j) = (L_{x_i}+L_{x_j})K_t(x_i,x_j)$. Moreover the Pfaffian $\text{Pf}(K^{(2n)}(t,\mathbf{x}))$ is a sum of terms each of product form

$$K_t(x_{\pi_1},x_{\pi_2})\dots K_t(x_{\pi_{2n-1}},x_{\pi_{2n}}) \tag{3.13}$$

for some permutation π, containing each of the variables $(x_i : i \le 2n)$ exactly once. Hence by the product rule, each term, and therefore the entire Pfaffian, solves the desired equation $\partial_t u = \sum_{i=1}^{2n} L_{x_i} u$ when $\mathbf{x} \in V_{2n}$.

Note that $|K_t(x,y)| \le 1$ and hence the Pfaffian $\text{Pf}(K^{(2n)}(t,\mathbf{x}))$ is a uniformly bounded function on $[0,\infty)\times V_{2n}$ (by (2.6), for example). Uniqueness of solutions to the sequence $((\text{ODE})_{2n} : n = 1,2,\dots)$ now implies that

$$\mathbb{E}_\eta\left[\Sigma_{\mathbf{x}}(\eta_t)\right] = u^{(2n)}(t,\mathbf{x}) = \text{Pf}(K^{(2n)}(t,\mathbf{x})), \quad \text{for } n \ge 1, t \ge 0 \text{ and } \mathbf{x} \in V_{2n}.$$

The lemma under consideration states that this identity holds also for $\mathbf{x} \in \overline{V}_{2n}$. However, by repeating the argument for the boundary conditions, for $\mathbf{x} \in \overline{V}_{2n}$ any equalities in $x_1 \le x_2 \le \cdots \le x_{2n}$ can be removed, pair by pair, until

$$u^{(2n)}(t,\mathbf{x}) = u^{(2m)}(t,\mathbf{z}), \qquad \text{and} \qquad \text{Pf}(K^{(2n)}(t,\mathbf{x})) = \text{Pf}(K^{(2m)}(t,\mathbf{z}))$$

for some $m \le n$ and $\mathbf{z} \in V_{2m}$, and hence equality also holds on the larger set. □

Remark 5. Averaging (3.12) over configurations gives

$$\mathbb{E}_{\eta_0}\left[\Sigma_{\mathbf{x}}(\eta_t)\right] = \mathbb{E}_{\eta_0}\left[\text{Pf}(\sigma_{x_i,x_j}(\eta_t) : i < j \le 2n)\right].$$

Hence we may interpret lemma 2 as an interchange of expectation and Pfaffian.

Proof of theorem 1. The particle intensities $\mathbb{E}[\eta_t(x_1)\dots\eta_t(x_n)]$ can be recovered from the product spin expectations. Indeed

$$\begin{aligned} D_y^+\sigma_{x,y}(\eta) &= \sigma_{x,y+1}(\eta) - \sigma_{x,y}(\eta) \\ &= \sigma_{x,y}(\eta)\left((-\theta)^{\eta(y)} - 1\right) \\ &= -(1+\theta)\,\eta(y)\,\sigma_{x,y}(\eta), \end{aligned}$$

so that

$$D_y^+\sigma_{x,y}(\eta)\big|_{y=x} = -(1+\theta)\eta(x). \tag{3.14}$$

Thus out of a spin pair, we can reconstruct a single occupancy variable by first a discrete derivative, and then an evaluation. We now iterate this to get at multiple occupancy variables. Fix $n \geq 1$ and consider $\mathbf{x} = (x_1, \hat{x}_1, \ldots, x_n, \hat{x}_n) \in \overline{V}_{2n}$ where

$$x_1 \leq \hat{x}_1 < x_2 \leq \hat{x}_2 < x_3 \leq \cdots < x_n \leq \hat{x}_n.$$

The restriction that $\hat{x}_i < x_{i+1}$ allows us to apply the operators $D^+_{\hat{x}_1}, \ldots, D^+_{\hat{x}_n}$ to both sides of the identity (from Lemma 2)

$$\mathbb{E}_\eta \left[\Sigma_{\mathbf{x}}(\eta_t)\right] = \mathrm{Pf}(K^{(2n)}(t, \mathbf{x})).$$

The left hand side becomes

$$(-1)^n(1+\theta)^n \mathbb{E}_\eta \left[\eta_t(x_1) \ldots \eta_t(x_n) \prod_{i=1}^n \sigma_{x_i, \hat{x}_i}(\eta_t)\right].$$

After setting $x_i = \hat{x_i}$ for all i we reach the intensity $(-1)^n(1+\theta)^n \mathbb{E}_\eta[\eta_t(x_1) \ldots \eta_t(x_n)]$. Applying the operators $D^+_{\hat{x}_1}, \ldots, D^+_{\hat{x}_n}$ to the Pfaffian on the right hand side preserves the Pfaffian structure. Indeed applying $D^+_{\hat{x}_1}$ to the single Pfaffian term (3.13) will change only a single factor in the product, namely

$$K_t(x_{\pi_{2i-1}}, x_{\pi_{2i}}) \to \begin{cases} D_1^+ K_t(x_{\pi_{2i-1}}, x_{\pi_{2i}}) & \text{if } \pi_{2i-1} = 1, \\ D_2^+ K_t(x_{\pi_{2i-1}}, x_{\pi_{2i}}) & \text{if } \pi_{2i} = 1. \end{cases}$$

The terms can then be summed into a new Pfaffian where the entries in the second row and column, which are the only entries containing the variable $\hat{x}_1$, are changed. Repeating this for the operators $D^+_{\hat{x}_2}, \ldots, D^+_{\hat{x}_n}$ and then setting $x_i = \hat{x}_i$ for all i, we still have a Pfaffian, and incorporating the factor $(-1)^n(1+\theta)^n$ the final entries are given exactly by the kernel $\mathbf{K}$ stated in the theorem. □

Remark 6. For random initial conditions the point process η_t will not in general be a Pfaffian point process, although by conditioning on the initial condition, the intensities can always be written as the expectation of a Pfaffian. However, under some random initial conditions, η_t does remain a Pfaffian point process. Indeed, examining the proof of Lemma 2, one needs only that the expectation $\mathbb{E}\left[\Sigma_{\mathbf{x}}(\eta_0)\right]$, for $\mathbf{x} \in V_{2n}$, can be written as a Pfaffian $\mathrm{Pf}(\Phi(x_i, x_j) : i < j \leq 2n)$ for some $\Phi : V_2 \to \mathbb{R}$. One then replaces the initial condition in the equation (3.8) for $K_t(x, y)$ by $\Phi(x, y)$ and the rest of the argument goes through. An important example is independent sites $(\eta_0(x) : x \in \mathbb{Z})$ with $\eta_0(x)$ a Bernoulli(λ_x) variable for some $\lambda_x \in [0, 1]$. Indeed for $\mathbf{x} \in V_{2n}$ and $x_0 < x_1$, one may use the Pfaffian quotient formula (proposition 9)

with $a_i = \prod_{z=x_0}^{x_i-1}(1-\lambda_z(1+\theta))^{-1}$ to write

$$\mathbb{E}\left[\Sigma_{\mathbf{x}}(\eta_0)\right] = \prod_{i=1}^{n}\frac{a_{2i-1}}{a_{2i}} = \mathrm{Pf}\left(\prod_{z\in[x_i,x_j)}(1-(1+\theta)\lambda_z) : i<j\leq 2n\right).$$

Remark 7. A slightly more combinatorial way of writing out the argument for the last part of the proof of theorem 1 is as follows. Starting from (3.9) we may reconstruct the product intensities as

$$\begin{aligned}\mathbb{E}\left[\prod_{i=1}^{n}\eta_t(x_i)\right] &= \mathbb{E}\left[\prod_{i=1}^{n}\frac{(1-\sigma_{x_i,x_i+1}(\eta_t))}{1+\theta}\right] \\ &= (1+\theta)^{-n}\sum_{m=0}^{n}(-1)^m\sum_{\substack{y_1<\cdots<y_m\\ \in\{x_1,\ldots,x_n\}}}\mathbb{E}\left[\prod_{i=1}^{m}\sigma_{y_i,y_i+1}(\eta_t)\right]. \qquad (3.15)\end{aligned}$$

Since the vector $y^{(2m)} = (y_1, y_1+1, \ldots, y_m, y_m+1) \in \overline{V}_{2m}$ we may apply Lemma 2 to see that

$$\mathbb{E}\left[\prod_{i=1}^{n}\eta_t(x_i)\right] = (1+\theta)^{-n}\sum_{m=0}^{n}(-1)^m\sum_{\substack{y_1<\cdots<y_m\\ \in\{x_1,\ldots,x_n\}}}\mathrm{Pf}\left(K^{(2m)}(t,y^{(2m)})\right).$$

By the formula for the Pfaffian of a sum (see proposition 4) this may be recombined as the single Pfaffian

$$\mathbb{E}\left[\prod_{i=1}^{n}\eta_t(x_i)\right] = (-1)^n(1+\theta)^{-n}\,\mathrm{Pf}\left(K^{(2n)}(t,y^{(2n)}) - J_{2n}\right),$$

where J_{2n} is the block diagonal matrix formed by n copies of $\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$. This shows that η_t is a Pfaffian point process with the kernel $\tilde{\mathbf{K}}$ given, for $x<y$, by

$$\hat{\mathbf{K}}(x,y) = \frac{-1}{1+\theta}\begin{pmatrix}K_t(x,y) & K_t(x,y+1)\\ K_t(x+1,y) & K_t(x+1,y+1)\end{pmatrix},$$

and $\tilde{\mathbf{K}}_{12}(x,x) = \frac{-1}{1+\theta}(K_t(x,x+1)-1)$, and other entries determined by anti-symmetry. Finally, the desired kernel $\mathbf{K}$ is obtained from $\hat{\mathbf{K}}$ by applying the elementary transform of subtracting the first row and column from the second row and column. By proposition 12 (part 3) the kernels $\hat{\mathbf{K}}$ and $\mathbf{K}$ determine the same point process.

Example 7. We conclude this section by putting theorem 1 into action through a

concrete example. Consider symmetric CARW on $\mathbb{Z}$ with product Bernoulli initial condition. The process is defined by setting $q_x = p_x = 1$ and $\eta_0(x)$ to be independent Bernoulli(λ) for all $x \in \mathbb{Z}$ and some $\lambda \in [0,1]$. Note that the $\lambda = 1$ case is the discrete analogue of example 6. The one-particle generator is the discrete Laplacian $\Delta = D^+ + D^-$. Theorem 1 ensures that the system is Pfaffian with kernel (3.6) given in terms of $K_t(x,y)$, solving

$$\begin{cases} \partial_t K_t(x,y) &= (\Delta_x + \Delta_y) K_t(x,y) & \text{for } x < y,\ t > 0, \\ K_t(x,x) &= 1 & \text{for all } x,\ t > 0, \\ K_0(x,y) &= (1 - (1+\theta)\lambda)^{y-x} & \text{for } x \le y. \end{cases} \tag{3.16}$$

The discrete two-dimensional heat kernel $p_t : \mathbb{Z}^2 \to [0,\infty)$ is given by

$$p_t(x,y) = e^{-4t} I_x(2t) I_y(2t), \tag{3.17}$$

where $I_x(t) = \frac{1}{\pi}\int_0^\pi e^{t\cos(w)}\cos(xw)\,\mathrm{d}w$ is the modified Bessel function of the first kind. Using linearity to force Dirichlet boundary conditions and recasting from $\{x < y\}$ to $\mathbb{Z}^2$ by the method of images, the ODE (3.16) may be solved explicitly in terms of p_t, namely

$$K_t(x,y) = 1 + \sum_{w_1,w_2 \in \mathbb{Z}} p_t(x - w_1, y - w_2)\left((1-(1+\theta)\lambda)^{|w_2 - w_1|} - 1\right)\operatorname{sgn}(w_2 - w_1),$$

for $x \le y$. We return to this example in chapter 4, where we give the asymptotics under diffusive scaling.

3.3 Models with branching or immigration

Two generalisations of CARW, in the pure interaction regimes of ARW and CRW, are shown to be Pfaffian point processes for a wide class of initial conditions, including deterministic. The models are annihilating random walks with pairwise immigration, and coalescing random walks with branching. Besides extending the theory, these models arise in various contexts.

Annihilating random walks with pairwise immigration. The Ising model of equilibrium statistical mechanics was introduced by Lenz and Ising as a model of ferromagnetism in metals. The model is a random assignment of ± 1 spin values to each site of $\mathbb{Z}$. Glauber [27] proposed viewing the Ising model as the large-time equilibrium limit of an evolution of spin configurations. Under the so-called *Glauber*

dynamics, spin values independently flip according to rates determined by nearest neighbour spins. Glauber established an expression for the interaction intensity in terms of the constants: magnetisation, temperature and the Boltzmann constant. This model has since become one of the most popular in non-equilibrium statistical mechanics, as it many cases it can be solved explicitly and important system characteristics written in terms of special functions [9, 21, 28, 41].

Within a spin configuration, the right-most site in a block of aligned spins is called a *domain wall*. Under the dynamics of the chain, the domain walls form a system of annihilating random walks on $\mathbb{Z}$ [3, 44]. As mentioned in remark 4, a spatially inhomogeneous version of the Glauber dynamics leads to the domain walls being described by the purely annihilating random walk model of section 3.2. These dynamics are said to be at zero temperature because the spins only move to increase local alignment and have no 'thermal energy' to flip of their own volition. The positive temperature Glauber dynamics allow for spins to flip regardless of their neighbours. In this case the domain walls still form an annihilating system, however the event of a spin flipping out of local alignment represents creation of two adjacent domain walls. Thus a system of annihilating random walks with pairwise immigration is embedded in the Ising spin chain under positive temperature Glauber dynamics.

Branching coalescing random walks. The mechanisms of branching and coalescence are mainstays in a variety of research areas. In population dynamics, for example, branching coalescing random walks can effectively model growth and propagation of a collection of organisms. Indeed migration is accounted for by the random walk component, births are represented by branching and deaths by coalescence, where deaths are interpreted to occur due to local overpopulation. Such systems are also known as reaction-diffusion models and may be used to describe chemical processes as well as those appearing in physics and biology.

Besides the rich applied literature, there is an another important motivation with roots in stochastic processes. The Brownian web is a continuum system of instantly coalescing Brownian motions started from all points in time and space, which arises as the limit of various discrete models. It was introduced by Arratia [5] and subsequently developed by various authors, see [22, 47, 56] and references therein. It is shown in [59] that at any given time the distribution of points on paths starting from the origin is a Pfaffian point process. A related object is the Brownian net [47, 55],

a continuum process involving branching coalescing Brownian motions. There is an open question in [47] to identify Pfaffian structure in the point set of the Brownian net. Branching coalescing random walks are a discrete-space counterpart that can be scaled to obtain continuum processes.

Despite the unconnected model mechanics and motivations, we will show that annihilating random walks with pairwise immigration and branching coalescing random walks are related, for certain initial conditions, via a generalised thinning relation involving thickening.

3.3.1 Annihilating random walks with pairwise immigration

A class of annihilating random walk systems on $\mathbb{Z}$ with pairwise immigration are shown to be Pfaffian point processes for a large set of initial conditions, including deterministic.

The model is a generalisation of the ARW model in section 3.2. We explain the dynamics informally before explicitly defining the model via its generator. Between interactions all particles jump independently following a nearest neighbour random walk on $\mathbb{Z}$, jumping

$$x \to x-1 \text{ at rate } q_x, \qquad \text{and} \qquad x-1 \to x \text{ at rate } p_x.$$

Independently there is

$$\text{immigration on sites } \{x-1,\, x\} \text{ at rate } m_x.$$

Immigration respects annihilation: if a particle immigrates onto an occupied site then the existing and new particles instantaneously annihilate.

These dynamics are encoded in the generator, which, for suitable $F : \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$, is given by

$$\mathcal{L}^A F(\eta) = \sum_{x\in\mathbb{Z}} q_x \left(F(\eta_{x,x-1}) - F(\eta)\right) + \sum_{x\in\mathbb{Z}} p_x \left(F(\eta_{x-1,x}) - F(\eta)\right) + \sum_{x\in\mathbb{Z}} m_x \left(F(\eta^i_{x-1,x}) - F(\eta)\right), \quad (3.18)$$

where $\eta_{x,y}$ (resp. $\eta^i_{x,y}$) is the new configuration resulting from a jump from x to y (resp. immigration onto x and y)

$$\begin{cases} \eta_{x,y}(z) = \eta^i_{x,y}(z) = \eta(z) & \text{for } z \notin \{x,y\}, \\ \eta_{x,y}(x) = 0, & \\ \eta_{x,y}(y) = \eta(x) + \eta(y) \mod 2, & \\ \eta^i_{x,y}(z) = 1 - \eta(z) & \text{for } z \in \{x,y\}. \end{cases}$$

We take $(p_x, q_x, m_x \geq 0 : x \in \mathbb{Z})$ uniformly bounded, then the generator defines the law of a unique Markov process, for any initial condition $\eta_0 \in \{0,1\}^{\mathbb{Z}}$, which we refer to as *annihilating random walks with pairwise immigration (ARWI)*. We denote its law by $\mathbb{P}_{\eta_0}$ and $\mathbb{E}_{\eta_0}$ on path space with canonical variables $\{\eta_t : t \geq 0\}$.

There is a bijection between the collection of ARWI models and domain walls for Ising spin chains under inhomogeneous positive temperature Glauber dynamics.

Using the notation (3.3), we set $K_t(x,y)$ to be the spin expectation

$$K_t(x,y) = \mathbb{E}_{\eta_0}\left[(-1)^{\eta_t[x,y)}\right], \qquad t \geq 0,\ x,y \in \mathbb{Z} \text{ with } x \leq y. \tag{3.19}$$

We can now state the main result for ARWI.

Theorem 2. *For any initial condition $\eta_0 \in \{0,1\}^{\mathbb{Z}}$, and at any fixed time $t \geq 0$, the ARWI variable η_t is a Pfaffian point process with kernel $\mathbf{K}$ given, for $x < y$, by*

$$\mathbf{K}(x,y) = -\frac{1}{2}\begin{pmatrix} K_t(x,y) & D_2^+ K_t(x,y) \\ D_1^+ K_t(x,y) & D_1^+ D_2^+ K_t(x,y) \end{pmatrix}, \tag{3.20}$$

and $\mathbf{K}_{12}(x,x) = -\frac{1}{2} D_2^+ K_t(x,x)$, with other entries determined by anti-symmetry.

As in the case of ARW, the spin expectation for ARWI may be characterised by a differential equation. We define a one-particle generator L^A, acting on $f : \mathbb{Z} \to \mathbb{R}$ by

$$L^A f(x) = q_x D^+ f(x) + p_x D^- f(x) - 2m_x f(x). \tag{3.21}$$

We will show that the spin expectation $(K_t(x,y) : t \geq 0, x, y \in \mathbb{Z}, x < y)$ is the unique bounded solution to the equation

$$\begin{cases} \partial_t K_t(x,y) &= (L^A_x + L^A_y) K_t(x,y) & \text{for } x < y,\ t > 0, \\ K_t(x,x) &= 1 & \text{for all } x,\ t > 0, \\ K_0(x,y) &= (-1)^{\eta_0[x,y)} & \text{for } x \leq y. \end{cases} \tag{3.22}$$

Note that L^A differs from the one-particle generator L for ARW (3.7) only through the presence of an additional potential term, accounting for immigration.

Proof outline. The proof of theorem 2 is a small generalisation to theorem 1 for ARW. Indeed the spin products $\eta \mapsto (-1)^{\eta[x_1,x_2)} \dots (-1)^{\eta[x_{2n-1},x_{2n})}$ for $n \geq 1$ are again suitable Markov duality functions, see lemma 3, with one-particle generator L^A. It then suffices to check that each step of the proof in section 3.2.2 extends to one-particle generators with a potential term.

Theorem 2 also holds for certain random initial conditions, including when $(\eta_0(x) : x \in \mathbb{Z})$ are independent. This follows from the same reasoning as for theorem 1, explained in remark 6.

Proof of theorem 2. For $n \geq 1$ and $\mathbf{x} = (x_1, \dots, x_{2n})$ with $x_1 \leq \dots \leq x_{2n}$ the spin product is given by

$$\Sigma_{\mathbf{x}}(\eta) = \prod_{i=1}^{n} (-1)^{\eta[x_{2i-1},x_{2i})}.$$

Note that $\Sigma_{\mathbf{x}}(\eta)$ only depends on a finite number of configuration sites and so lies in the domain of the generator $\mathcal{L}^A$. The heart of the proof is in showing that the spin products $\Sigma_{\mathbf{x}}(\eta)$ remain suitable Markov duality functions for $\mathcal{L}^A$. This duality is encoded in the following computation.

Lemma 3. *For $\mathbf{x} = (x_1, \dots, x_{2n})$ with $x_1 < \dots < x_{2n}$ the action of the particle generator $\mathcal{L}^A$ on $\Sigma_{\mathbf{x}}(\eta)$ is*

$$\mathcal{L}^A \Sigma_{\mathbf{x}}(\eta) = \sum_{i=1}^{n} L^A_{x_i} \Sigma_{\mathbf{x}}(\eta).$$

Proof of lemma 3. From lemma 1, the part of $\mathcal{L}^A \Sigma_{\mathbf{x}}(\eta)$ due to particle motion is given by $\sum_{i=1}^{2n} L_{x_i} \Sigma_{\mathbf{x}}(\eta)$. It remains to compute the contribution from immigration, namely

$$\sum_{z \in \mathbb{Z}} m_z \left(\Sigma_{\mathbf{x}}(\eta^i_{z-1,z}) - \Sigma_{\mathbf{x}}(\eta) \right).$$

Note that the modified configuration for immigration $\eta^i_{z,z+1}$, differs from η on at most two sites, z and $z+1$. Since the entries of $\mathbf{x}$ are strictly ordered, the intervals $[x_{2i-1}, x_{2i})$ are separated by at least one site, whereby adjacent sites $z, z+1$ can intersect at most one of these intervals. In particular the value of at most one of

the spins $(-1)^{\eta[x_{2i-1},x_{2i})}$ will change in the event of an immigration. This allows us to separate the immigration contribution to $\mathcal{L}^A$ using

$$\begin{aligned}&\Sigma_{\mathbf{x}}(\eta^i_{z,z+1}) - \Sigma_{\mathbf{x}}(\eta)\\ &\qquad = \sum_{i=1}^{n}\left(\prod_{j\neq i}(-1)^{\eta[x_{2j-1},x_{2j})}\right)\left((-1)^{\eta^i_{z,z+1}[x_{2i-1},x_{2i})} - (-1)^{\eta[x_{2i-1},x_{2i})}\right). \quad (3.23)\end{aligned}$$

Fix $x < y$ and consider the contribution for a single spin product $(-1)^{\eta[x,y)}$, namely

$$\sum_{z\in\mathbb{Z}} m_z\left((-1)^{\eta^i_{z-1,z}[x,y)} - (-1)^{\eta[x,y)}\right).$$

The terms indexed by $z \leq x-1$ and $z \geq y+1$ are zero, as the modified configuration in these cases is unchanged in the interval $[x, y)$. The terms $x+1 \leq z \leq y-1$ are also zero, since

$$\begin{aligned}&(-1)^{\eta^i_{z-1,z}[x,y)} - (-1)^{\eta[x,y)}\\ &\qquad = \Big(\prod_{\substack{w=x\\ w\neq z-1,z}}^{y-1}(-1)^{\eta(w)}\Big)\left((-1)^{1-\eta(z-1)}(-1)^{1-\eta(z)} - (-1)^{\eta(z-1)}(-1)^{\eta(z)}\right) = 0.\end{aligned}$$

The remaining terms give identical non-zero contributions. For example, when $z = x$ the computation is

$$\begin{aligned}(-1)^{\eta^i_{x-1,x}[x,y)} - (-1)^{\eta[x,y)} &= \Big(\prod_{w=x+1}^{y-1}(-1)^{\eta(w)}\Big)\left((-1)^{1-\eta(x)} - (-1)^{\eta(x)}\right)\\ &= -2(-1)^{\eta[x,y)}.\end{aligned}$$

The case $z = y$ is similar. Using (3.23) the immigration term is given by

$$\sum_{z\in\mathbb{Z}} m_z\left(\Sigma_{\mathbf{x}}(\eta^i_{z-1,z}) - \Sigma_{\mathbf{x}}(\eta)\right) = -2\Sigma_{\mathbf{x}}(\eta)\sum_{i=1}^{2n} m_{x_i}.$$

Collecting the jump and immigration terms gives the desired expression

$$\mathcal{L}^A\Sigma_{\mathbf{x}}(\eta) = \sum_{i=1}^{2n} L_{x_i}\Sigma_{\mathbf{x}}(\eta) - 2\Sigma_{\mathbf{x}}(\eta)\sum_{i=1}^{2n} m_{x_i} = \sum_{i=1}^{2n} L^A_{x_i}\Sigma_{\mathbf{x}}(\eta).$$

□

Following the proof outline for theorem 1, the duality of lemma 3 allows us (via (3.11)) to recast the Kolmogorov equation for the product spin expectation as an ODE built from L^A. We check that generalising the one-particle generator does not destroy the scalar Pfaffian solution.

Lemma 4. *For all $\eta_0 \in \{0,1\}^{\mathbb{Z}}$, for all $n \geq 1$, $\mathbf{x} = (x_1, \dots, x_{2n})$ with $x_1 \leq \cdots \leq x_{2n}$ and $t \geq 0$*

$$\mathbb{E}_{\eta_0}\left[\Sigma_{\mathbf{x}}(\eta_t)\right] = \mathrm{Pf}(K^{(2n)}(t,\mathbf{x})), \tag{3.24}$$

where $K^{(2n)}(t,\mathbf{x})$ is the anti-symmetric $2n \times 2n$ matrix with entries $K_t(x_i, x_j)$ for $i < j$, defined by (3.19).

Proof of lemma 4. The result is an extension to lemma 2 for more general one-particle generators and the proof is identical. We give a brief outline of the steps. Using the ODE framework of lemma 2, for $n \geq 1$ the ARWI product spin expectation $u^{(2n)}(t,\mathbf{x}) = \mathbb{E}_{\eta_0}\left[\Sigma_{\mathbf{x}}(\eta_t)\right]$ solves

$$(\mathrm{ODE})^A_{2n} \qquad \begin{cases} \partial_t u^{(2n)}(t,\mathbf{x}) &=& \sum_{i=1}^{2n} L^A_{x_i} u^{(2n)}(t,\mathbf{x}) & \text{on } [0,\infty) \times V_{2n}, \\ u^{(2n)}(t,\mathbf{x}) &=& u^{(2n-2)}(t,\mathbf{x}^{i,i+1}) & \text{on } [0,\infty) \times \partial V^{(i)}_{2n}, \\ u^{(2n)}(0,\mathbf{x}) &=& \Sigma_{\mathbf{x}}(\eta_0) & \text{on } V_{2n}, \end{cases}$$

with $u^{(0)} = 1$. The infinite sequence of equations $((\mathrm{ODE})^A_{2n} : n = 1, 2, \dots)$ are uniquely solvable, within the class of continuously differentiable functions that are uniformly bounded on $[0,\infty) \times V_{2n}$ (see appendix A for details). It remains to check that the sequence of scalar Pfaffians $(\mathrm{Pf}(K^{(2n)}(t,\mathbf{x})) : n = 1, 2, \dots)$ is a uniformly bounded solution. The verification of initial conditions, boundary conditions and the differential equation pass through unchanged from the proof of lemma 2. Moreover the Pfaffian is uniformly bounded on $[0,\infty) \times V_{2n}$, whereby uniqueness gives the desired equality on V_{2n}. Finally the solution may be extended to $\overline{V}_{2n}$ by sequentially removing coincidental points. □

Proof of theorem 2. All that remains is to recover the particle intensities from the product spin expectations. The steps in the proof of theorem 1 apply to any point process whose spin product expectations are given by a scalar Pfaffian. In particular the particle intensities are Pfaffian with kernel of the form (3.6) (with $\theta = 1$). This completes the proof of theorem 2. □

Example 8. We apply the theorem to a simple example. Consider annihilating symmetric random walks on $\mathbb{Z}$ with homogeneous pairwise immigration and product Bernoulli initial condition. This is the case $q_x = p_x = 1$, $m_x = m$ and $\eta_0(x)$ is

Bernoulli(λ) for all $x \in \mathbb{Z}$ and some $m > 0$, $\lambda \in [0, 1]$. The one-particle generator is $L^A = \Delta - 2m$. By theorem 2 the position of particles at time $t \geq 0$ is Pfaffian with kernel (3.20), solving

$$\begin{cases} \partial_t K_t(x,y) &= (\Delta_x + \Delta_y) K_t(x,y) - 4m K_t(x,y) & \text{for } x < y,\ t > 0, \\ K_t(x,x) &= 1 & \text{for all } x,\ t > 0, \\ K_0(x,y) &= (1-2\lambda)^{y-x} & \text{for } x \leq y. \end{cases} \tag{3.25}$$

Building on example 7 and using Duhamel's principle to handle the additional driving term (of the equation forced to have Dirichlet boundary conditions), the explicit solution is given by

$$\begin{aligned} K_t(x,y) = 1 &+ e^{-4mt} \sum_{w_1, w_2 \in \mathbb{Z}} p_t(x - w_1, y - w_2) \left((1-2\lambda)^{|w_2 - w_1|} - 1 \right) \operatorname{sgn}(w_2 - w_1) \\ &- 4m \int_0^t e^{-4ms} \sum_{w_1, w_2 \in \mathbb{Z}} p_s(x - w_1, y - w_2) \operatorname{sgn}(w_2 - w_1) \, \mathrm{d}s, \end{aligned} \tag{3.26}$$

for $x \leq y$, where p_t is the discrete two-dimensional heat kernel (3.17). More details are given in section 4.3.1, where we consider the continuum scaling limit.

Remark 8. By setting the immigration rate to zero, $m_x = 0$ for all $x \in \mathbb{Z}$, the ARW model with rates $(p_x, q_x \geq 0 : x \in \mathbb{Z})$ is recovered.

3.3.2 Branching coalescing random walks

A class of coalescing random walk systems on $\mathbb{Z}$ with branching are shown to be Pfaffian point processes for a large set of initial conditions, including deterministic.

The model is a generalisation of the CRW model in section 3.2. We explain the dynamics informally before explicitly defining the model via its generator. Between interactions particles perform independent nearest neighbour random walks with jumps

$$x \to x - 1 \text{ at rate } q_x, \qquad \text{and} \qquad x - 1 \to x \text{ at rate } p_x.$$

If a particle jumps onto an occupied site then the two particles instantaneously coalesce. Independently a particle branches

$$x \to \{x - 1, x\} \text{ at rate } \ell_x, \qquad \text{and} \qquad x - 1 \to \{x - 1, x\} \text{ at rate } r_x.$$

Branching events respect coalescence: if a particle branches onto an occupied site then the existing and new particles instantaneously coalesce.

The generator for these dynamics is given, for suitable $F : \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$, by

$$\mathcal{L}^C F(\eta) = \sum_{z\in\mathbb{Z}} q_x \left(F(\eta_{z,z-1}) - F(\eta)\right) + \sum_{z\in\mathbb{Z}} p_x \left(F(\eta_{z-1,z}) - F(\eta)\right) + \sum_{z\in\mathbb{Z}} \ell_x \left(F(\eta^b_{z,z-1}) - F(\eta)\right) + \sum_{z\in\mathbb{Z}} r_x \left(F(\eta^b_{z-1,z}) - F(\eta)\right), \quad (3.27)$$

where $\eta_{x,y}$ (resp. $\eta^b_{x,y}$) is the new configuration resulting from a jump (resp. branching) from x to y

$$\begin{cases} \eta_{x,y}(z) = \eta^b_{x,y}(z) = \eta(z) \quad \text{for } z \notin \{x,y\}, \\ \eta_{x,y}(x) = 0, \\ \eta^b_{x,y}(x) = \eta(x), \\ \eta_{x,y}(y) = \eta^b_{x,y}(y) = \min\{1, \eta(x) + \eta(y)\}. \end{cases}$$

We take $(p_x, q_x, \ell_x, r_x \geq 0 : x \in \mathbb{Z})$ to be uniformly bounded, then the generator defines the law of a unique Markov process, for any initial condition $\eta_0 \in \{0,1\}^{\mathbb{Z}}$, which we refer to as *branching coalescing random walks (BCRW)*. We denote its law by $\mathbb{P}_{\eta_0}$ and $\mathbb{E}_{\eta_0}$ on path space with canonical variables $\{\eta_t : t \geq 0\}$.

Using the notation (3.3), we set $K_t(x,y)$ to be the empty interval probability

$$K_t(x,y) = \mathbb{E}_{\eta_0}\left[1\left(\eta_t[x,y) = 0\right)\right], \qquad t \geq 0,\ x,y \in \mathbb{Z} \text{ with } x \leq y. \quad (3.28)$$

In order for BCRW to be Pfaffian we impose the following conditions on the rates

$$\frac{\ell_x}{q_x} = \frac{r_{x+1}}{p_{x+1}} < M, \quad (3.29a)$$

$$c = r_x - \ell_x, \quad (3.29b)$$

for all $x \in \mathbb{Z}$ and some M, $c \in \mathbb{R}$. An interpretation of the conditions is discussed in remark 11. Under condition (3.29a), we may define a bounded function $\phi : \mathbb{Z} \to [1,\infty)$ by

$$\phi_x = \sqrt{1 + \frac{\ell_x}{q_x}} = \sqrt{1 + \frac{r_{x+1}}{p_{x+1}}}. \quad (3.30)$$

We now set

$$\tilde{K}_t(x,y) = K_t(x,y) \prod_{z=x}^{y-1} \phi_z, \qquad \text{for } t \geq 0,\ x,y \in \mathbb{Z} \text{ with } x \leq y, \quad (3.31)$$

using the convention that the empty product is equal to 1, so that $\tilde{K}_t(x,x) = 1$.

Theorem 3. *Let* (3.29) *be satisfied. For any initial condition* $\eta_0 \in \{0,1\}^{\mathbb{Z}}$*, and at any fixed time* $t \geq 0$*, the BCRW variable* η_t *is a Pfaffian point process with kernel* $\tilde{\mathbf{K}}$ *given, for* $x < y$*, by*

$$\tilde{\mathbf{K}}(x,y) = -\frac{1}{\sqrt{\phi_x \phi_y}} \begin{pmatrix} \tilde{K}_t(x,y) & D_2^+ \tilde{K}_t(x,y) \\ D_1^+ \tilde{K}_t(x,y) & D_1^+ D_2^+ \tilde{K}_t(x,y) \end{pmatrix}, \tag{3.32}$$

and $\tilde{\mathbf{K}}_{12}(x,x) = 1 - \frac{1}{\phi_x} \tilde{K}_t(x,x+1)$*, with other entries determined by anti-symmetry.*

The technical origins of ϕ_x and the conditions (3.29) in the context of the underlying proof method are discussed in remark 12 after the proof of theorem 3.

As for CRW, the BCRW empty interval probability $K_t(x,y)$ may be characterised by a differential equation. We define one-particle generators $L^{C,(1)}$ and $L^{C,(2)}$, acting on $f : \mathbb{Z} \to \mathbb{R}$, by

$$L^{C,(1)} f(x) = q_x D^+ f(x) + p_x D^- f(x) + r_x D^- f(x), \tag{3.33}$$

$$L^{C,(2)} f(x) = q_x D^+ f(x) + p_x D^- f(x) + \ell_x D^+ f(x). \tag{3.34}$$

We will see that the empty interval probability $(K_t(x,y) : t \geq 0, x, y \in \mathbb{Z}, x < y)$ solves

$$\begin{cases} \partial_t K_t(x,y) &= (L_x^{C,(1)} + L_y^{C,(2)}) K_t(x,y) & \text{for } x < y,\ t > 0, \\ K_t(x,x) &= 1 & \text{for all } x,\ t > 0, \\ K_0(x,y) &= 1\,(\eta_0[x,y) = 0) & \text{for } x \leq y. \end{cases} \tag{3.35}$$

Looking forward, this already shows us that the empty interval indicators $\eta \mapsto 1(\eta[x_1,x_2) = \cdots = \eta[x_{2n-1},x_{2n}) = 0)$ for $n \geq 1$ are not suitable Markov duality functions for BCRW, because even for $n = 1$ the resultant ODEs distinguish odd- and even-indexed sites. Indeed, for our proof method, we are aiming for a scalar Pfaffian expression for the duality expectations, whose corresponding ODEs would be in terms of a single one-particle operator applied to all variables x_i (see the proof of lemma 2). One way to get around this is to introduce a judicious choice of multiplicative factor to the empty interval indicators, forcing the resultant ODEs to have the desired form. This is the origin of the ϕ_x factor (3.30). The price we pay for suitable ODEs is that the modified empty interval indicators are no longer bounded, however they are of exponential growth. We define a one-particle

generator $\tilde{L}^C$, acting on $f : \mathbb{Z} \to \mathbb{R}$, by

$$\tilde{L}^C f(x) = q_x \phi_x D^+ f(x) + p_x \phi_{x-1} D^- f(x) - \left(\frac{q_x}{2}(1-\phi_x)^2 + \frac{p_x}{2}(1-\phi_{x-1})^2 \right) f(x). \tag{3.36}$$

We will show that, under conditions (3.29), the modified empty interval probability $(\tilde{K}_t(x,y) : t \geq 0, x, y \in \mathbb{Z}, x < y)$ is the unique solution to the equation

$$\begin{cases} \partial_t \tilde{K}_t(x,y) &= (\tilde{L}^C_x + \tilde{L}^C_y)\tilde{K}_t(x,y) & \text{for } x < y,\ t > 0, \\ \tilde{K}_t(x,x) &= 1 & \text{for all } x,\ t > 0, \\ \tilde{K}_0(x,y) &= 1\left(\eta_0[x,y) = 0\right) \prod_{z=x}^{y-1} \phi_z & \text{for } x \leq y, \end{cases} \tag{3.37}$$

within the class of functions of exponential growth.

Remark 9. Note that the kernel $\tilde{\mathbf{K}}$ for BCRW at time $t \geq 0$ is locally bounded, since by condition (3.29a)

$$|\tilde{K}_t(x,y)| \leq (1+M)^{|y-x|/2},$$

so that $\tilde{\mathbf{K}}$ satisfies

$$\max_{i,j \in \{1,2\}} |\tilde{\mathbf{K}}_{i,j}(x,y)| \leq 4(1+M)^{(|y-x|+1)/2}.$$

Proposition 13 guarantees that $\tilde{\mathbf{K}}$ determines the Pfaffian point process.

Remark 10. As usual the result extends to certain random initial conditions, following the reasoning of remark 6. One example is independent Bernoulli(λ) initial conditions for $\lambda \in (0,1]$. In this case the initial condition of (3.36) is given by $\tilde{K}_0(x,y) = \prod_{z=x}^{y-1}(1-\lambda)\phi_z$. Note that $\tilde{K}_0(x,y)$, and hence $\tilde{K}_t(x,y)$, is uniformly bounded provided $\lambda \geq 1 - 1/\phi_z$ for all $z \in \mathbb{Z}$. By condition (3.29a) this holds for $\lambda \geq 1 - (1+M)^{-1/2}$.

Remark 11. The rate condition (3.29a) may be interpreted as a kind of local flux conversation. Indeed the ratio ℓ_x/q_x comprises rates for leftward jump and branching events from the site $x \in \mathbb{Z}$, and this is balanced by the analogous rightward quantity r_{x+1}/p_{x+1} for site x. The conditions simplify in the homogeneous case $p_x = p > 0$, $q_x = q > 0$, $\ell_x = \ell$ and $r_x = r$ for all $x \in \mathbb{Z}$. In particular condition (3.29b) is automatically satisfied and therefore obsolete. The rate conditions reduce to $\ell/q = r/p$. In section 4.4 we develop diffusive scaling limits for BCRW and we will see that this condition actually represents symmetry of the branching rates (see remark 20).

Proof outline. As indicated above, establishing theorem 3 requires a modification to the proof of theorem 1 for CRW. Indeed the empty interval indicators $\eta \mapsto 1(\eta[x_1, x_2) = \cdots = \eta[x_{2n-1}, x_{2n}) = 0)$ for $n \geq 1$ are not suitable Markov duality functions for $\mathcal{L}^C$, because the dual action splits between two one-particle generators, as shown in lemma 5. Imposing conditions (3.29) and introducing ϕ_x, the modified empty interval indicators

$$\eta \mapsto 1(\eta[x_1, x_2) = \cdots = \eta[x_{2n-1}, x_{2n}) = 0) \prod_{i=1}^{n} \prod_{z=x_{2i-1}}^{x_{2i}-1} \phi_z,$$

for $n \geq 1$ have the desired action under $\mathcal{L}^C$, as shown in lemma 6. It follows that the modified empty interval probabilities

$$\mathbb{E}_{\eta_0} \left[1(\eta[x_1, x_2) = \cdots = \eta[x_{2n-1}, x_{2n}) = 0)\right] \prod_{i=1}^{n} \prod_{z=x_{2i-1}}^{x_{2i}-1} \phi_z,$$

for $n \geq 1$ are solutions of $2n$-dimensional ODEs built from a single one-particle generator. We are now in familiar territory. The ODEs are of the right form to be solved exactly by Pfaffians built from the scalar kernel $\tilde{K}_t(x, y)$, as shown in lemma 7.

It remains to recover the particle intensities from the modified empty interval probabilities. Dividing the modified empty interval probabilities and the Pfaffian through by the ϕ_x product and passing the factors onto the kernel entries, the empty interval probabilities are given by Pfaffians with a 2×2 block structure. As in the proof of theorem 1, we can reconstruct particle intensities from empty interval probabilities by discrete derivatives. The Pfaffian and 2×2 block structure are preserved, leading to the desired kernel $\tilde{\mathbf{K}}$.

Proof of theorem 3. For $n \geq 1$ and $\mathbf{x} = (x_1, \ldots, x_{2n})$ with $x_1 \leq \cdots \leq x_{2n}$ we define the empty interval product

$$\Sigma_{\mathbf{x}}(\eta) = \prod_{i=1}^{n} 1(\eta[x_{2i-1}, x_{2i}) = 0),$$

and its modification

$$\tilde{\Sigma}_{\mathbf{x}}(\eta) = \Sigma_{\mathbf{x}}(\eta)\Phi_{\mathbf{x}}, \tag{3.38}$$

where the modification factor is given by

$$\Phi_{\mathbf{x}} = \prod_{i=1}^{n} \prod_{z=x_{2i-1}}^{x_{2i}-1} \phi_z. \tag{3.39}$$

Since $\Sigma_{\mathbf{x}}(\eta)$ and $\tilde{\Sigma}_{\mathbf{x}}(\eta)$ only depend on finitely many sites of η, both lie in the domain of the generator $\mathcal{L}^C$. The proceeding computations determine the suitability of $\Sigma_{\mathbf{x}}(\eta)$ and $\tilde{\Sigma}_{\mathbf{x}}(\eta)$ as Markov duality functions.

Lemma 5. *For $\mathbf{x} = (x_1, \ldots, x_{2n})$ with $x_1 < \cdots < x_{2n}$ the action of the particle generator $\mathcal{L}^C$ on $\Sigma_{\mathbf{x}}(\eta)$ is*

$$\mathcal{L}^C \Sigma_{\mathbf{x}}(\eta) = \sum_{i=1}^{n} (L^{C,(1)}_{x_{2i-1}} + L^{C,(2)}_{x_{2i}}) \Sigma_{\mathbf{x}}(\eta).$$

Lemma 6. *For $\mathbf{x} = (x_1, \ldots, x_{2n})$ with $x_1 < \cdots < x_{2n}$ the action of the particle generator $\mathcal{L}^C$ on $\tilde{\Sigma}_{\mathbf{x}}(\eta)$ is*

$$\mathcal{L}^C \tilde{\Sigma}_{\mathbf{x}}(\eta) = \sum_{i=1}^{2n} \tilde{L}^C_{x_i} \tilde{\Sigma}_{\mathbf{x}}(\eta).$$

Proof of lemma 5. From lemma 1 for CRW, the part of $\mathcal{L}^C \Sigma_{\mathbf{x}}(\eta)$ due to particle motion is given by $\sum_{i=1}^{2n} L_{x_i} \Sigma_{\mathbf{x}}(\eta)$, where L is defined by (3.7). It remains to compute the contribution from branching. Consider the left branching term

$$\sum_{z \in \mathbb{Z}} \ell_z \left(\Sigma_{\mathbf{x}}(\eta^b_{z,z-1}) - \Sigma_{\mathbf{x}}(\eta) \right).$$

The modified configuration due to left branching $\eta^b_{z,z-1}$ differs from η on at most one site, $z-1$, so for each $z \in \mathbb{Z}$ there will be a change in at most one of the indicators $1(\eta[x_{2i-1}, x_{2i}) = 0)$. This allows us to separate the left branching contribution to $\mathcal{L}^C$ using

$$\begin{aligned} &\Sigma_{\mathbf{x}}(\eta^b_{z,z-1}) - \Sigma_{\mathbf{x}}(\eta) \\ &= \sum_{i=1}^{n} \Big(\prod_{j \neq i} 1(\eta[x_{2i-1}, x_{2i}) = 0) \Big) \Big(1(\eta^b_{z,z-1}[x_{2i-1}, x_{2i}) = 0) - 1(\eta[x_{2i-1}, x_{2i}) = 0) \Big). \end{aligned} \tag{3.40}$$

Fix $x < y$ and consider the contribution for a single empty interval indicator

$1(\eta[x,y)=0)$, namely

$$\sum_{z\in\mathbb{Z}} \ell_z \left(1(\eta^b_{z,z-1}[x,y)=0) - 1(\eta[x,y)=0)\right).$$

The terms indexed by $z \le x$ and $z \ge y+1$ are zero, as the modified configuration in these cases is unchanged in the interval $[x,y)$. The terms $x+1 \le z \le y-1$ are also zero, since for the empty interval indicator to differ on $\eta^b_{z,z-1}$ and η, there must be a particle at z to branch to the left from, in which case both indicators are zero. The remaining term, $z=y$, follows from a simple computation

$$\begin{aligned} 1(\eta^b_{y,y-1}[x,y)=0) - 1(\eta[x,y)=0) &= 1(\eta[x,y+1)=0) - 1(\eta[x,y)=0) \\ &= D^+_y 1(\eta[x,y)=0). \end{aligned}$$

Using (3.40), the left branching term is given by

$$\sum_{z\in\mathbb{Z}} \ell_z \left(\Sigma_{\mathbf{x}}(\eta^b_{z,z-1}) - \Sigma_{\mathbf{x}}(\eta)\right) = \sum_{i=1}^{n} \ell_{x_{2i}} D^+_{x_{2i}} \Sigma_{\mathbf{x}}(\eta).$$

A similar calculation reveals that the contribution from right branching is

$$\sum_{z\in\mathbb{Z}} r_z \left(\Sigma_{\mathbf{x}}(\eta^b_{z-1,z}) - \Sigma_{\mathbf{x}}(\eta)\right) = \sum_{i=1}^{n} r_{x_{2i-1}} D^-_{x_{2i-1}} \Sigma_{\mathbf{x}}(\eta).$$

Collecting the jump and branching terms gives the desired expression

$$\begin{aligned} \mathcal{L}^C \Sigma_{\mathbf{x}}(\eta) &= \sum_{i=1}^{2n} L_{x_i} \Sigma_{\mathbf{x}}(\eta) + \sum_{i=1}^{n} \ell_{x_{2i}} D^+_{x_{2i}} \Sigma_{\mathbf{x}}(\eta) + \sum_{i=1}^{n} r_{x_{2i-1}} D^-_{x_{2i-1}} \Sigma_{\mathbf{x}}(\eta) \\ &= \sum_{i=1}^{n} (L^{C,(1)}_{x_{2i-1}} + L^{C,(2)}_{x_{2i}}) \Sigma_{\mathbf{x}}(\eta). \end{aligned}$$

□

Proof of lemma 6. We prove the result by direct calculation. Since $\Phi_{\mathbf{x}}$ is independent of η, lemma 5 gives

$$\mathcal{L}^C \tilde{\Sigma}_{\mathbf{x}}(\eta) = \Phi_{\mathbf{x}} \mathcal{L}^C \Sigma_{\mathbf{x}}(\eta) = \Phi_{\mathbf{x}} \sum_{i=1}^{n} L^{C,(1)}_{x_{2i-1}} \Sigma_{\mathbf{x}}(\eta) + \Phi_{\mathbf{x}} \sum_{i=1}^{n} L^{C,(2)}_{x_{2i}} \Sigma_{\mathbf{x}}(\eta). \tag{3.41}$$

For both terms, we expand and massage into the form of an operator applied to $\Phi_{\mathbf{x}} \Sigma_{\mathbf{x}}(\eta)$. For the sake of clarity it is convenient in the calculations to replace $\Sigma_{\mathbf{x}}(\eta)$

by a function of one argument, $f : \mathbb{Z} \to \mathbb{R}$. Expanding the $L^{C,(1)}_{x_{2i-1}}$ term

$$\Phi_{\mathbf{x}} L^{C,(1)}_{x_{2i-1}} f(x_{2i-1}) = \Phi_{\mathbf{x}} q_{x_{2i-1}} f(x_{2i-1}+1) + \Phi_{\mathbf{x}}(p_{x_{2i-1}} + r_{x_{2i-1}}) f(x_{2i-1}-1) \\ - \Phi_{\mathbf{x}}(q_{x_{2i-1}} + p_{x_{2i-1}} + r_{x_{2i-1}}) f(x_{2i-1}).$$

We develop the first two terms separately, in each case applying the decomposition $\Phi_{\mathbf{x}} = \Phi_{\mathbf{x}^{2i-1,2i}} \prod_{z=x_{2i-1}}^{x_{2i}-1} \phi_z$. For the first term

$$\Phi_{\mathbf{x}} q_{x_{2i-1}} f(x_{2i-1}+1) = q_{x_{2i-1}} \phi_{x_{2i-1}} \Phi_{\mathbf{x}^{2i-1,2i}} \Big(\prod_{z=x_{2i-1}+1}^{x_{2i}-1} \phi_z \Big) f(x_{2i-1}+1).$$

For the second term

$$\Phi_{\mathbf{x}}(p_{x_{2i-1}} + r_{x_{2i-1}}) f(x_{2i-1}-1) = \frac{p_{x_{2i-1}} + r_{x_{2i-1}}}{\phi_{x_{2i-1}-1}} \Phi_{\mathbf{x}^{2i-1,2i}} \Big(\prod_{z=x_{2i-1}-1}^{x_{2i}-1} \phi_z \Big) f(x_{2i-1}-1).$$

Using (3.30) the ratio may be expressed as

$$\frac{p_x + r_x}{\phi_{x-1}} = p_x \phi_{x-1}. \tag{3.42}$$

All together, for an odd-indexed variable $x = x_{2i-1}$, the $L^{C,(1)}_x$ term may be written as an operator applied to $\Phi_{\mathbf{x}} f(x)$

$$\Phi_{\mathbf{x}} L^{C,(1)}_x f(x) = q_x \phi_x D^+_x \left(\Phi_{\mathbf{x}} f(x)\right) + p_x \phi_{x-1} D^-_x \left(\Phi_{\mathbf{x}} f(x)\right) \\ - \left(q_x(1-\phi_x) + p_x(1-\phi_{x-1}) + r_x\right) \Phi_{\mathbf{x}} f(x). \tag{3.43}$$

A similar calculation for the $L^{C,(2)}_x$ term, with an even-indexed variable $x = x_{2i}$, gives

$$\Phi_{\mathbf{x}} L^{C,(2)}_x f(x) = q_x \phi_x D^+_x \left(\Phi_{\mathbf{x}} f(x)\right) + p_x \phi_{x-1} D^-_x \left(\Phi_{\mathbf{x}} f(x)\right) \\ - \left(q_x(1-\phi_x) + p_x(1-\phi_{x-1}) + \ell_x\right) \Phi_{\mathbf{x}} f(x), \tag{3.44}$$

where we use (3.30) to write

$$\frac{q_x + \ell_x}{\phi_x} = q_x \phi_x. \tag{3.45}$$

The operators on the right-hand sides of (3.43) and (3.44) still differ. The coefficients of the discrete derivative terms coincide and match $\tilde{L}^C$, but those of the potential term do not. Note that by using (3.29a) the potential coefficient of $\tilde{L}^C$ may be

rewritten as

$$\frac{q_x}{2}(1-\phi_x)^2+\frac{p_x}{2}(1-\phi_{x-1})^2=q_x(1-\phi_x)+p_x(1-\phi_{x-1})+\frac{r_x+\ell_x}{2}.$$

Moreover condition (3.29b) allows us to substitute

$$r_x=\frac{r_x+\ell_x}{2}+\frac{c}{2}\qquad\text{and}\qquad \ell_x=\frac{r_x+\ell_x}{2}-\frac{c}{2}, \tag{3.46}$$

into (3.43) and (3.44), respectively, giving

$$\begin{aligned}\Phi_{\mathbf{x}}L_x^{C,(1)}f(x_{2i-1})&=\tilde{L}^C(\Phi_{\mathbf{x}}f(x_{2i-1}))+\frac{c}{2}\Phi_{\mathbf{x}}f(x_{2i-1}),\\ \Phi_{\mathbf{x}}L_x^{C,(2)}f(x_{2i})&=\tilde{L}^C(\Phi_{\mathbf{x}}f(x_{2i}))-\frac{c}{2}\Phi_{\mathbf{x}}f(x_{2i}).\end{aligned}$$

Returning to the function $\tilde{\Sigma}_{\mathbf{x}}(\eta)$, (3.41) becomes

$$\mathcal{L}^C\tilde{\Sigma}_{\mathbf{x}}(\eta)=\sum_{i=1}^{n}\left(\tilde{L}^C_{x_{2i-1}}\tilde{\Sigma}_{\mathbf{x}}(\eta)+\frac{c}{2}\tilde{\Sigma}_{\mathbf{x}}(\eta)\right)+\sum_{i=1}^{n}\left(\tilde{L}^C_{x_{2i}}\tilde{\Sigma}_{\mathbf{x}}(\eta)-\frac{c}{2}\tilde{\Sigma}_{\mathbf{x}}(\eta)\right).$$

The c terms cancel leaving the claimed formula in terms of a single generator. □

The duality of lemma 6 allows us (via (3.11)) to recast the Kolmogorov equation for the modified empty interval probabilities as an ODE built from $\tilde{L}^C$. We check that the scalar Pfaffian is a solution to this ODE system.

Lemma 7. *For all $\eta_0\in\{0,1\}^{\mathbb{Z}}$, for all $n\geq 1$, $\mathbf{x}=(x_1,\ldots,x_{2n})$ with $x_1\leq\cdots\leq x_{2n}$ and $t\geq 0$*

$$\mathbb{E}_{\eta_0}\left[\tilde{\Sigma}_{\mathbf{x}}(\eta_t)\right]=\mathrm{Pf}(\tilde{K}^{(2n)}(t,\mathbf{x})),$$

where $\tilde{K}^{(2n)}(t,\mathbf{x})$ is the anti-symmetric $2n\times 2n$ matrix with entries $\tilde{K}_t(x_i,x_j)$ for $i<j$, defined by (3.31).

Proof of lemma 7. The result is the analogue for BCRW of lemma 2 and we follow the same proof outline. For $n\geq 1$ the modified empty interval probability $u^{(2n)}(t,\mathbf{x})=\mathbb{E}_{\eta_0}\left[\tilde{\Sigma}_{\mathbf{x}}(\eta_t)\right]$ solves

$$(\mathrm{ODE})^C_{2n}\qquad\begin{cases}\partial_t u^{(2n)}(t,\mathbf{x}) &=& \sum_{i=1}^{2n}\tilde{L}^C_{x_i}u^{(2n)}(t,\mathbf{x}) & \text{on } [0,\infty)\times V_{2n},\\ u^{(2n)}(t,\mathbf{x}) &=& u^{(2n-2)}(t,\mathbf{x}^{i,i+1}) & \text{on } [0,\infty)\times\partial V^{(i)}_{2n},\\ u^{(2n)}(0,\mathbf{x}) &=& \tilde{\Sigma}_{\mathbf{x}}(\eta_0) & \text{on } V_{2n},\end{cases}$$

with $u^{(0)}=1$. The only step to check is the boundary condition. Note that $\Phi_{\mathbf{x}}$

may be pulled outside the expectation, $u^{(2n)}(t,\mathbf{x}) = \Phi_{\mathbf{x}}\mathbb{E}_{\eta_0}\left[\Sigma_{\mathbf{x}}(\eta_t)\right]$. The boundary condition for $u^{(2n)}(t,\mathbf{x})$ follows from the analogous boundary conditions for the empty interval probability $\mathbb{E}_{\eta_0}\left[\Sigma_{\mathbf{x}}(\eta_t)\right]$, which we have already seen in lemma 2, and for $\Phi_{\mathbf{x}}$. The boundary condition for $\Phi_{\mathbf{x}}$ is immediate from its product form (3.39). Note that $\Phi_{\mathbf{x}}$ is not bounded, however by (3.29a)

$$\Phi_{\mathbf{x}} \leq (1+M)^{\sum_{i=1}^{n}(x_{2i}-x_{2i-1})/2},$$

so $u^{(2n)}(t,\mathbf{x})$ has exponential growth of rate $\frac{1}{2}\ln(1+M)$.

The infinite sequence of equations $((\mathrm{ODE})_{2n}^{C} : n = 1, 2, \dots)$ are uniquely solvable, within the class of continuously differentiable functions on $[0,\infty) \times V_{2n}$ that have exponential growth of rate $\frac{1}{2}\ln(1+M)$ (see appendix A for details). In fact for convenience we note that each equation $(\mathrm{ODE})_{2n}^{C}$ has a unique solution in the larger space of functions with exponential growth of rate $\frac{n}{2}\ln(1+M) \geq \frac{1}{2}\ln(1+M)$. It remains to check that the sequence of scalar Pfaffians $(\mathrm{Pf}(\tilde{K}^{(2n)}(t,\mathbf{x})) : n = 1, 2, \dots)$ is a solution with the claimed exponential growth. The one-particle generator $\tilde{L}^{C}$ has the same form as the generator L^{A} for ARWI, and as in the analogous ARWI result, lemma 4, the verification of the differential equation passes through unchanged from the proof of lemma 2. The boundary condition check also translates directly, as all the proof relies on is the Pfaffian structure and the $n = 1$ boundary condition $\tilde{K}_t(x,y) = 1$ for $x = y$. For the initial conditions, fix $\mathbf{x} \in V_{2n}$ and choose $x_0 \leq x_1$. The entries of the Pfaffian at time zero can be rewritten as

$$\tilde{K}_0(x_i,x_j) = \lim_{\theta\downarrow 0} \frac{(-\theta)^{\eta_0[x_0,x_j)}}{(-\theta)^{\eta_0[x_0,x_i)}} \frac{\prod_{z=x_0}^{x_j-1}\phi_z}{\prod_{z=x_0}^{x_i-1}\phi_z}.$$

Applying proposition 9 to the pre-limiting $\theta > 0$ entries, and then letting $\theta \downarrow 0$, gives

$$\mathrm{Pf}(\tilde{K}^{(2n)}(0,\mathbf{x})) = \prod_{i=1}^{n} \frac{1(\eta_0[x_0,x_{2i-1}) = 0)}{1(\eta_0[x_0,x_{2i}) = 0)} \frac{\prod_{z=x_0}^{x_{2i-1}-1}\phi_z}{\prod_{z=x_0}^{x_{2i-1}-1}\phi_z} = \tilde{\Sigma}_{\mathbf{x}}(\eta_0).$$

Finally, by (3.29a) and the penultimate bound of (2.6), the Pfaffian satisfies

$$|\,\mathrm{Pf}(\tilde{K}^{(2n)}(t,\mathbf{x}))| \leq (1+M)^{n(x_n-x_1)/2}(2n)^{n/2}, \qquad \text{for } \mathbf{x} \in V_{2n},$$

so has exponential growth of rate $\frac{n}{2}\ln(1+M)$. (It is this bound that dictates the above choice of function space.) The claimed equality on V_{2n} follows by uniqueness. Finally, the result extends to $\overline{V}_{2n}$ by the standard method of removing coincidental points. □

Proof of theorem 3. It remains to recover the particle intensities from the modified empty interval probabilities. Pulling the modification factor $\Phi_{\mathbf{x}}$ outside the expectation, lemma 7 gives

$$\mathbb{E}_{\eta_0}\left[\tilde{\Sigma}_{\mathbf{x}}(\eta_t)\right] = \Phi_{\mathbf{x}}\mathbb{E}_{\eta_0}\left[\Sigma_{\mathbf{x}}(\eta_t)\right] = \mathrm{Pf}(\tilde{K}^{(2n)}(t,\mathbf{x})).$$

Dividing both sides through by $\Phi_{\mathbf{x}}$, the empty interval probabilities are given by

$$\mathbb{E}_{\eta_0}\left[\Sigma_{\mathbf{x}}(\eta_t)\right] = \mathrm{Pf}(\tilde{K}^{(2n)}(t,\mathbf{x}))\Phi_{\mathbf{x}}^{-1}.$$

The outline is to pass the $\Phi_{\mathbf{x}}$ factor onto the kernel, apply derivatives to obtain intensities, and then pull the remaining ϕ_x terms outside the Pfaffian. Fix $\mathbf{x}$ and choose $x_0 \leq x_1$. The factor $\Phi_{\mathbf{x}}^{-1}$ may be written as the determinant of a $2n \times 2n$ diagonal matrix B with entries

$$B_{2i-1,2i-1} = \prod_{z=x_0}^{x_{2i-1}-1} \phi_z, \qquad B_{2i,2i} = \prod_{z=x_0}^{x_{2i}-1} \frac{1}{\phi_z}.$$

Applying the conjugation formula (proposition 3) to $B^T\tilde{K}^{(2n)}(t,\mathbf{x})B$, the empty interval probabilities are themselves Pfaffian

$$\mathbb{E}_{\eta_0}\left[\Sigma_{\mathbf{x}}(\eta_t)\right] = \mathrm{Pf}(\hat{K}^{(2n)}(t,\mathbf{x})),$$

where $\hat{K}^{(2n)}(t,\mathbf{x})$ is the anti-symmetric $2n \times 2n$ matrix in 2×2 block form with the general $(2i-1,2i) \times (2j-1,2j)$ block, for $i < j$, given by

$$\begin{pmatrix} \prod_{z=x_0}^{x_{2i-1}-1} \phi_z \prod_{z=x_0}^{x_{2j-1}-1} \phi_z \, \tilde{K}_t(x_{2i-1}, x_{2j-1}) & \prod_{z=x_{2i-1}}^{x_{2j}-1} \frac{1}{\phi_z} \, \tilde{K}_t(x_{2i-1}, x_{2j}) \\ \prod_{z=x_{2i}}^{x_{2j-1}-1} \phi_z \, \tilde{K}_t(x_{2i}, x_{2j-1}) & \prod_{z=x_0}^{x_{2i}-1} \frac{1}{\phi_z} \prod_{z=x_0}^{x_{2j}-1} \frac{1}{\phi_z} \, \tilde{K}_t(x_{2i}, x_{2j}) \end{pmatrix},$$

and $\left(\hat{K}^{(2n)}(t,\mathbf{x})\right)_{2i-1,2i} = \prod_{z=x_{2i-1}}^{x_{2i}-1} \frac{1}{\phi_z} \tilde{K}_t(x_{2i-1}, x_{2i})$. As in the proof of theorem 1, the particle intensities are recovered by applying discrete derivatives and evaluations to the empty interval probabilities. These operations may be passed onto the entries, preserving the 2×2 block structure and leading to the kernel $\mathbf{K}^{(1)}(x,y)$ for the

particle intensities, given, for $x_0 \leq x < y$, by

$$\begin{pmatrix} -\prod_{z=x_0}^{x-1}\phi_z \prod_{z=x_0}^{y-1}\phi_z\,\tilde{K}_t(x,y) & -D_y^+\left(\prod_{z=x}^{y-1}\frac{1}{\phi_z}\,\tilde{K}_t(x,y)\right) \\ -D_x^+\left(\prod_{z=x}^{y-1}\phi_z\,\tilde{K}_t(x,y)\right) & -D_x^+D_y^+\left(\prod_{z=x_0}^{x-1}\frac{1}{\phi_z}\prod_{z=x_0}^{y-1}\frac{1}{\phi_z}\,\tilde{K}_t(x,y)\right) \end{pmatrix},$$

and $\mathbf{K}_{12}^{(1)}(x,x) = -D_y^+\left(\prod_{z=x}^{y-1}\frac{1}{\phi_z}\,\tilde{K}_t(x,y)\right)\big|_{y=x}$. We now perform manipulations to obtain the desired kernel $\tilde{\mathbf{K}}$. First we expand the discrete derivatives to give

$$\begin{aligned}
\mathbf{K}_{11}^{(1)}(x,y) &= -\prod_{z=x_0}^{x-1}\phi_z \prod_{z=x_0}^{y-1}\phi_z\,\tilde{K}_t(x,y);\\
\mathbf{K}_{12}^{(1)}(x,y) &= -\textstyle\prod_{z=x}^{y-1}\frac{1}{\phi_z}\left(\frac{1}{\phi_y}\tilde{K}_t(x,y+1)-\tilde{K}_t(x,y)\right);\\
\mathbf{K}_{21}^{(1)}(x,y) &= -\textstyle\prod_{z=x}^{y-1}\phi_z\left(\frac{1}{\phi_x}\tilde{K}_t(x+1,y)-\tilde{K}_t(x,y)\right);\\
\mathbf{K}_{22}^{(1)}(x,y) &= -\prod_{z=x_0}^{x-1}\frac{1}{\phi_z}\prod_{z=x_0}^{y-1}\frac{1}{\phi_z}\left(\frac{1}{\phi_x\phi_y}\tilde{K}_t(x+1,y+1)\right.\\
&\qquad \left.-\frac{1}{\phi_x}\tilde{K}_t(x+1,y)-\frac{1}{\phi_y}\tilde{K}_t(x,y+1)+\tilde{K}_t(x,y)\right),
\end{aligned}$$

and $\mathbf{K}_{12}^{(1)}(x,x) = 1-\frac{1}{\phi_x}\tilde{K}_t(x,x+1)$. We can now identify $\mathbf{K}^{(1)}$ as an inhomogeneous shift, as in proposition 12 part 2, of the equivalent kernel $\mathbf{K}^{(2)}$ with the function $f(x) = \prod_{z=x_0}^{x-1}\phi_z$, given, for $x < y$, by

$$\begin{aligned}
\mathbf{K}_{11}^{(2)}(x,y) &= -\tilde{K}_t(x,y);\\
\mathbf{K}_{12}^{(2)}(x,y) &= -\tfrac{1}{\phi_y}\tilde{K}_t(x,y+1)+\tilde{K}_t(x,y);\\
\mathbf{K}_{21}^{(2)}(x,y) &= -\tfrac{1}{\phi_x}\tilde{K}_t(x+1,y)+\tilde{K}_t(x,y);\\
\mathbf{K}_{22}^{(2)}(x,y) &= -\tfrac{1}{\phi_x\phi_y}\tilde{K}_t(x+1,y+1)\\
&\qquad +\tfrac{1}{\phi_x}\tilde{K}_t(x+1,y)+\tfrac{1}{\phi_y}\tilde{K}_t(x,y+1)-\tilde{K}_t(x,y),
\end{aligned}$$

and $\mathbf{K}_{12}^{(2)}(x,x) = 1-\frac{1}{\phi_x}\tilde{K}_t(x,x+1)$. We remark that transforming from $\mathbf{K}^{(1)}$ to $\mathbf{K}^{(2)}$ is achieved by conjugating with a determinant one diagonal matrix $\hat{B}$ with entries

$$\hat{B}_{2i-1,2i-1} = \prod_{z=x_0}^{x_{2i-1}-1}\phi_z, \qquad \hat{B}_{2i,2i} = \prod_{z=x_0}^{x_{2i-1}-1}\frac{1}{\phi_z}.$$

Note that the matrix $\hat{B}$ is the same as B under the evaluations to derive particle intensities, and in this way, the transformation can be interpreted as undoing the original conjugation by B after obtaining intensities. Continuing, $\mathbf{K}^{(2)}$ is itself a transformation of a simpler kernel. Indeed proposition 12 part 3 (with $c = 1$)

implies that an equivalent alternative kernel is given, for $x < y$, by

$$\mathbf{K}^{(3)}(x,y) = -\begin{pmatrix} \tilde{K}_t(x,y) & \frac{1}{\phi_y}\tilde{K}_t(x,y+1) \\ \frac{1}{\phi_x}\tilde{K}_t(x+1,y) & \frac{1}{\phi_x\phi_y}\tilde{K}_t(x+1,y+1) \end{pmatrix},$$

and $\mathbf{K}^{(3)}_{12}(x,x) = 1 - \frac{1}{\phi_x}\tilde{K}_t(x,x+1)$. The ϕ_x factors may be tidied up by applying an inhomogeneous shift (proposition 12 part 2) with $f(x) = 1/\sqrt{\phi_x}$, giving the kernel

$$\mathbf{K}^{(4)}(x,y) = -\frac{1}{\sqrt{\phi_x\phi_y}}\begin{pmatrix} \tilde{K}_t(x,y) & \tilde{K}_t(x,y+1) \\ \tilde{K}_t(x+1,y) & \tilde{K}_t(x+1,y+1) \end{pmatrix},$$

and $\mathbf{K}^{(4)}_{12}(x,x) = 1 - \frac{1}{\phi_x}\tilde{K}_t(x,x+1)$. Finally, the desired kernel $\tilde{\mathbf{K}}$ is obtained from $\mathbf{K}^{(4)}$ by subtracting the first row and column from the second (proposition 12 part 3 with $c = -1$) in order to form discrete derivatives of $\tilde{K}_t(x,y)$. □

Remark 12. In (3.39) we chose the modification factor $\Phi_{\mathbf{x}} : V_{2n} \to \mathbb{R}$ for the empty interval indicators to be a product of ϕ_x factors, defined in (3.30). We show that these choices are natural for our purposes. For the modification to be useful, the ODEs for the modified probabilities should have Pfaffian solutions. In particular the modified probabilities, and hence $\Phi_{\mathbf{x}}$ itself, should solve the appropriate boundary conditions, namely

$$\Phi_{\mathbf{x}} = \Phi_{\mathbf{x}^{i,i+1}} \qquad \text{for } \mathbf{x} \in \partial V^{(i)}_{2n}, \tag{3.47}$$

with $\Phi_{(x,y)} = 1$ if $x = y$. Moreover the differential equation for the ODEs should be in terms of a single one-particle generator. As a first step, the action of the particle generator $\mathcal{L}^C$ on $\tilde{\Sigma}_{\mathbf{x}}(\eta)$ should decompose into a sum of (at this stage not even necessarily the same) one-particle generators, $\sum_{i=1}^{2n} L^{(i)}_{x_i}\tilde{\Sigma}_{\mathbf{x}}(\eta)$. This occurs provided

$$\frac{\Phi_{\mathbf{x}}}{\Phi_{\mathbf{x}^{i+}}} \quad \text{and} \quad \frac{\Phi_{\mathbf{x}}}{\Phi_{\mathbf{x}^{i-}}} \quad \text{are dependent only on } x_i, \tag{3.48}$$

for each $1 \le i \le 2n$, where $\mathbf{x}^{i\pm}$ is the vector $\mathbf{x}$ with the i-th variable incremented by ± 1. It can be proved by induction on n that (3.47) and (3.48) imply $\Phi_{\mathbf{x}}$ has the product form (3.39) for some $\phi : \mathbb{Z} \to \mathbb{R}$. The question is now when do the one-particle generators $L^{(i)}_x$ coincide. Each generator has the form $L^{(i)}_x f(x) = \alpha^{(i)}_x D^+ f(x) + \beta^{(i)}_x D^- f(x) + \gamma^{(i)}_x f(x)$. Provided $\Phi_{\mathbf{x}}$ is of product form, the discrete derivative coefficients coincide provided (3.42) and (3.45) are satisfied, which is equivalent to condition (3.29a) and definition (3.30) for ϕ_x. For the potential term, constant multiples may be redistributed evenly amongst the $L^{(i)}_{x_i}$, each receiving $\frac{1}{2n}\sum_{i=1}^{2n}\gamma^{(i)}_{x_i} f(\mathbf{x})$. Hence to obtain a single one-particle generator, the po-

tential term coefficients must not differ by more than a constant. Provided (3.29a) holds, this condition is equivalent to (3.29b). To summarise, within this framework, we do not believe that there are more general choices for $\Phi_{\mathbf{x}}$, that is, the conditions (3.29) characterise the richest BCRW structure.

Example 9. We consider a simple example of BCRW. Consider coalescing symmetric random walks on $\mathbb{Z}$ with homogeneous branching rates and product Bernoulli initial condition. This is the case $q_x = p_x = 1$, $\ell_x = r_x = b$ and $\eta_0(x)$ is Bernoulli(λ) for all $x \in \mathbb{Z}$ and some $b > 0$, $\lambda \in [0, 1]$. The modification factor is given by $\phi = \sqrt{1+b}$. The corresponding one-particle generator is $\tilde{L}^C = \phi\Delta - (1-\phi)^2$. Theorem 3 gives that the particle positions of BCRW with these rates at time $t \geq 0$ is Pfaffian with kernel (3.32), built from the function $\tilde{K}_t(x, y)$ solving

$$\begin{cases} \partial_t \tilde{K}_t(x,y) &= \phi(\Delta_x + \Delta_y)\tilde{K}_t(x,y) - 2(1-\phi)^2\tilde{K}_t(x,y) & \text{for } x < y,\ t > 0, \\ \tilde{K}_t(x,x) &= 1 & \text{for all } x,\ t > 0, \\ \tilde{K}_0(x,y) &= (\phi(1-\lambda))^{y-x} & \text{for } x \leq y. \end{cases}$$

Note that the ODE (3.25) for symmetric ARWI in example 8 has the same form. In particular, $\tilde{K}_t(x, y)$ may be expressed explicitly in terms of the discrete two-dimensional heat kernel

$$\begin{aligned} \tilde{K}_t(x,y) = 1 + e^{-2(1-\phi)^2 t} \sum_{w_1,w_2 \in \mathbb{Z}} p_t(x-w_1, y-w_2) \left((\phi(1-\lambda))^{|w_2-w_1|} - 1 \right) \operatorname{sgn}(w_2 - w_1) \\ - 2(1-\phi)^2 \int_0^t \sum_{w_1,w_2 \in \mathbb{Z}} p_s(x-w_1, y-w_2) \operatorname{sgn}(w_2 - w_1)\, \mathrm{d}s. \end{aligned}$$

The aforementioned ODE congruence forms the basis of a relation between BCRW and ARWI, explored further in section 3.3.3. We return to this example in chapter 4, where we investigate continuum scaling limits.

Remark 13. Upon setting the branching rates ℓ_x and r_x to zero for all $x \in \mathbb{Z}$, the core model of coalescing random walks with rates $(p_x, q_x > 0 : x \in \mathbb{Z})$ is recovered.

3.3.3 Relation between branching and immigration models

A relation between the one-dimensional distributions of ARWI and BCRW, under particular initial conditions, is derived. More precisely, we show that at a given time the distribution of BCRW particles suitably thinned is equal to the superposition of ARWI with an independent Bernoulli system. The basis for this connection is the observation that the defining ODEs for the kernel functions of BCRW and ARWI

have the same form, and can be shown to coincide for carefully chosen rates and initial conditions.

According to theorem 3, the kernel $\tilde{\mathbf{K}}$ for BCRW with rates $(q_x, p_x, \ell_x, r_x \geq 0 : x \in \mathbb{Z})$ satisfying (3.29) is given in terms of $\tilde{K}_t(x,y)$, characterised by an ODE built from the operator

$$\tilde{L}^C f(x) = q_x \phi_x D^+ f(x) + p_x \phi_{x-1} D^- f(x) - \Big(\frac{q_x}{2}(1-\phi_x)^2 + \frac{p_x}{2}(1-\phi_{x-1})^2 \Big) f(x),$$

where ϕ_x is given by (3.30). Note that the coefficient of the potential term is non-positive. On the other hand, theorem 2 gives the kernel $\mathbf{K}$ for ARWI with rates $(q_x, p_x, m_x \geq 0 : x \in \mathbb{Z})$ in terms of $K_t(x,y)$, determined by an ODE with operator

$$L^A f(x) = q_x D^+ f(x) + p_x D^- f(x) - 2m_x f(x).$$

The key observation is that the operators have the same form and coincide for a particular choice of rates. Indeed the one-particle generator for ARWI with rates $\big(q_x\phi_x, p_x\phi_{x-1}, \frac{q_x}{4}(1-\phi_x)^2 + \frac{p_x}{4}(1-\phi_{x-1})^2 : x \in \mathbb{Z}\big)$ is $\tilde{L}^C$. More precisely, under this choice, $K_t(x,y)$ solves

$$\begin{cases} \partial_t K_t(x,y) &= (\tilde{L}^C_x + \tilde{L}^C_y) K_t(x,y) & \text{for } x < y,\ t > 0, \\ K_t(x,x) &= 1 & \text{for all } x,\ t > 0, \\ K_0(x,y) &= (-1)^{\eta_0[x,y)} & \text{for } x \leq y. \end{cases}$$

The BCRW kernel function $\tilde{K}_t(x,y)$ solves

$$\begin{cases} \partial_t \tilde{K}_t(x,y) &= (\tilde{L}^C_x + \tilde{L}^C_y) \tilde{K}_t(x,y) & \text{for } x < y,\ t > 0, \\ \tilde{K}_t(x,x) &= 1 & \text{for all } x,\ t > 0, \\ \tilde{K}_0(x,y) &= 1\,(\eta_0[x,y) = 0) \prod_{z=x}^{y-1} \phi_z & \text{for } x \leq y. \end{cases}$$

If we can match the initial conditions then the ODEs coincide and we have a relation between the kernels for BCRW and ARWI. Recall that theorems 2 and 3 hold for certain random initial conditions, in particular when $(\eta_0(x) : x \in \mathbb{Z})$ are independent. Fix $\theta_x \in [0,1]$ for $x \in \mathbb{Z}$ and take $\eta_0(x)$ to be independent Bernoulli(θ_x). The corresponding initial conditions are given, for $K_t(x,y)$, by

$$K_0(x,y) = \mathbb{E}\left[\prod_{z=x}^{y-1} (-1)^{\eta_0(z)} \right] = \prod_{z=x}^{y-1} (1 - 2\theta_z),$$

and, for $\tilde{K}_t(x,y)$, by

$$\tilde{K}_0(x,y) = \mathbb{E}\left[\prod_{z=x}^{y-1} 1(\eta_0(z)=0)\phi_z\right] = \prod_{z=x}^{y-1}(1-\theta_z)\phi_z.$$

The expressions coincide upon taking the initial condition for ARWI to be independent Bernoulli(θ'_x) with $\theta'_x = \frac{1}{2}(1-\phi_x(1-\theta_x))$. We impose the condition $\theta_x \geq 1 - 1/\phi_x$, so that $\theta'_x \in [0,1]$. Note that this condition guarantees $\tilde{K}_t(x,y)$ is uniformly bounded, see remark 10. We henceforth consider BCRW with rates $(q_x, p_x, \ell_x, r_x \geq 0 : x \in \mathbb{Z})$ satisfying (3.29) and initial condition independent Bernoulli(θ_x) with $\theta_x \geq 1 - 1/\phi_x$, and ARWI with rates $(q_x\phi_x, p_x\phi_{x-1}, \frac{q_x}{2}(1-\phi_x)^2 + \frac{p_x}{2}(1-\phi_{x-1})^2 : x \in \mathbb{Z})$ and initial condition independent Bernoulli(θ'_x) with $\theta'_x = \frac{1}{2}(1-\phi_x(1-\theta_x))$, both at a fixed time $t \geq 0$. We have shown that $\tilde{K}_t(x,y) = K_t(x,y)$.

In order to develop the connection it is convenient to work with intensities. This is also justified as at a fixed time the ARWI and BCRW kernels are both locally bounded, so by proposition 13 the processes are determined by their intensities. The outline is to expand the BCRW intensities and identify the resulting expressions as intensities for a point process involving ARWI. By theorem 3 the particle intensity for BCRW at $\mathbf{x} = (x_1, \ldots, x_n)$ with $x_1 < \cdots < x_n$ is

$$\rho^{\mathrm{BCRW}_t}(\mathbf{x}) = \prod_{i=1}^{n} \frac{1}{\phi_{x_i}} \operatorname{Pf}\left(\hat{K}^{(2n)}(t,\mathbf{x})\right),$$

where $\hat{K}^{(2n)}(t,\mathbf{x})$ is the anti-symmetric $2n \times 2n$ matrix with 2×2 blocks given, for $i<j$, by

$$-\begin{pmatrix} \tilde{K}_t(x_i,x_j) & D_2^+\tilde{K}_t(x_i,x_j) \\ D_1^+\tilde{K}_t(x_i,x_j) & D_1^+D_2^+\tilde{K}_t(x_i,x_j) \end{pmatrix},$$

and, for $i=j$, by

$$\begin{pmatrix} 0 & \phi_{x_i} - \tilde{K}_t(x_i,x_i+1) \\ \tilde{K}_t(x_i,x_i+1) - \phi_{x_i} & 0 \end{pmatrix}.$$

The term $\prod_{i=1}^n \frac{1}{\phi_{x_i}}$ comes from pulling the factors $1/\sqrt{\phi_x\phi_y}$ of $\tilde{\mathbf{K}}$ outside the intensity Pfaffian, for example by conjugating with a diagonal matrix B with entries $B_{2i-1,2i-1} = B_{2i,2i} = \sqrt{\phi_{x_i}}$. Upon substituting $\tilde{K}_t(x,y) = K_t(x,y)$, the upper-triangular blocks of $\hat{K}^{(2n)}(t,\mathbf{x})$ are given exactly in terms of the ARWI kernel (3.20) by $2\mathbf{K}(x_i,x_j)$. Collecting the diagonal-block discrepancies into a separate matrix

$B(\mathbf{x})$, the particle intensity of BCRW is given by

$$\rho^{\mathrm{BCRW}_t}(\mathbf{x}) = \prod_{i=1}^{n} \frac{2}{\phi_{x_i}} \operatorname{Pf}\left(\mathbf{K}(\mathbf{x}) + \frac{1}{2}B(\mathbf{x})\right),$$

where $B(\mathbf{x})$ is a block-diagonal matrix generated by $\left(\begin{smallmatrix} 0 & \phi_{x_i}-1 \\ 1-\phi_{x_i} & 0 \end{smallmatrix}\right)$. By proposition 17 we may rewrite the right-hand side in terms of the intensity for the α-superposition of ARWI and Y, a system of independent Bernoulli($\lambda(x)$) random variables with rates $\lambda(x) \in [0,1]$ to be determined, namely

$$\rho^{\mathrm{BCRW}_t}(\mathbf{x}) = \prod_{i=1}^{n} \frac{2}{\phi_{x_i}} \frac{1}{1-(2-\alpha)\lambda(x_i)} \; \rho^{(\mathrm{ARWI}_t + Y)_\alpha}(\mathbf{x}).$$

In particular, the parameter $\mu(x)$ of proposition 17 is given by

$$\mu(x) = \frac{\lambda(x)}{1-(2-\alpha)\lambda(x)} = \frac{\phi_x - 1}{2}.$$

Note that $\mu(x) \in [0, \frac{1}{2}(\sqrt{1+M}-1)]$ by (3.29a), so that $1-(2-\alpha)\lambda(x) \neq 0$ and we are justified in applying (2.14). Solving for $\lambda(x)$ gives

$$\lambda(x) = \frac{\phi_x - 1}{(\phi_x - 1)(2-\alpha) + 2},$$

and we note that $\lambda(x) \in [0,1]$ for all $\phi_x \geq 1$ and $\alpha \in [0,1]$. The remaining product is given by

$$\prod_{i=1}^{n} \frac{2}{\phi_{x_i}} \frac{1}{1-(2-\alpha)\lambda(x_i)} = \prod_{i=1}^{n} \frac{(\phi_{x_i}-1)(2-\alpha)+2}{\phi_{x_i}} \geq 1,$$

with the inequality holding for all $\phi_x \geq 1$ and $\alpha \in [0,1]$. With proposition 16 in mind, we interpret an intensity multiplied by a product as inhomogeneous thinning. Dividing through by the product, so it takes values in $[0,1]$, we arrive at

$$\prod_{i=1}^{n} \frac{\phi_{x_i}}{(\phi_{x_i}-1)(2-\alpha)+2} \; \rho^{\mathrm{BCRW}_t}(\mathbf{x}) = \rho^{(\mathrm{ARWI}_t + Y)_\alpha}(\mathbf{x}). \tag{3.49}$$

We have proved the following result.

Theorem 4. *Fix $(q_x, p_x, \ell_x, r_x \geq 0 : x \in \mathbb{Z})$ satisfying (3.29), $\theta_x \in [1-1/\phi_x, 1]$, where ϕ_x is defined by (3.30), and $\alpha \in [0,1]$. Fix $t \geq 0$ and consider the following independent point processes on $\mathbb{Z}$:*

- BCRW_t, *BCRW at time* t *with rates* $(q_x, p_x, \ell_x, r_x : x \in \mathbb{Z})$ *and initial condition independent* $\mathrm{Bernoulli}(\theta_x)$*;*

- ARWI_t, *ARWI at time* t *with rates*

$$\left(q_x\phi_x, p_x\phi_{x-1}, \frac{q_x}{4}(1-\phi_x)^2 + \frac{p_x}{4}(1-\phi_{x-1})^2 : x \in \mathbb{Z}\right),$$

  and initial condition independent $\mathrm{Bernoulli}(\frac{1}{2}(1-\phi_x(1-\theta_x))$*;*

- Y*, a system of independent* $\mathrm{Bernoulli}\left(\frac{\phi_x - 1}{(\phi_x - 1)(2-\alpha)+2}\right)$ *random variables.*

Then the law of BCRW_t *with inhomogeneous thinning at rate* $\frac{\phi_x}{(\phi_x-1)(2-\alpha)+2}$ *is equal to the law of the* α*-superposition of* ARWI_t *and* Y*. In particular* (3.49) *holds for the intensities.*

As a consequence, we recover the well known thinning relation between CRW and ARW. By taking branching rates $\ell_x = r_x = 0$ for $x \in \mathbb{Z}$, we obtain CRW. In this case $\phi_x = 1$ and the ARWI immigration rate is zero, giving ARW. Moreover the Bernoulli rate for Y is zero, rendering the superposition redundant (since any point process is unchanged under α-superposition with the empty point process). Finally the thinning factor simplifies to $1/2$, both for the coalescing system and the initial condition of the annihilating system.

Corollary 1. *Fix* $\theta_x \in [0,1]$ *for* $x \in \mathbb{Z}$*. The law of thinned CRW at time* $t \geq 0$ *with independent* $\mathrm{Bernoulli}(\theta_x)$ *initial conditions is equal to the law of ARW at time* t *with independent* $\mathrm{Bernoulli}(\theta_x/2)$ *initial conditions.*

Chapter 4

Continuum scaling limits

We investigate continuum point processes by taking scaling limits of the discrete models developed in chapter 3. The limits remain Pfaffian and each kernel is built from the solution to a PDE and its derivatives. The key tool is a kernel continuity result, giving that Pfaffian point processes converge provided their corresponding kernels converge in an appropriate sense. For an interacting particle system this corresponds to convergence (of the one-dimensional marginals of the scaled processes) at a single (scaled) time. Each discrete kernel in chapter 3 is defined in terms of the solution to an ODE, and the kernel convergence amounts to checking uniform convergence of lattice approximations to a two-dimensional continuum PDE, at a fixed $t > 0$, along with their first and second derivatives.

We begin in section 4.1 with the convergence lemma for Pfaffian point processes. We then consider in turn the models of chapter 3, developing the limits of some spatially homogeneous models, namely with homogeneous rates and independent Bernoulli initial conditions. In particular, the corresponding limit PDEs may be solved explicitly in terms Gaussian integrals. For CARW the PDE convergence is shown to boil down to suitable convergence of the initial conditions and in section 4.2 we prove convergence of the scaled CARW models. The estimates do not apply to the ARWI and BCRW models, however we can still read off and solve the limit PDEs, characterising the limit point processes. The ARWI limits are considered in section 4.3, where we also consider the large-time limit. In section 4.4 we consider the more delicate case of BCRW, deriving the limit and obtaining the continuum analogue for scaling limits of the relation in section 3.3.3 between ARWI and BCRW.

Note that convergence results should hold for the full temporal discrete processes, but so far we have only described their one-dimensional marginals. We expect that in

each case there are underlying continuum stochastic processes with one-dimensional marginals given by the point process scaling limits we obtain. In chapter 5 we move in this direction by extending the Pfaffian structure to multiple times for ARWI.

4.1 Convergence of Pfaffian point processes via kernels

A kernel continuity result for sequences of Pfaffian point processes on $\epsilon\mathbb{Z}$ is proved, giving that convergence to a continuum point process follows from convergence of suitably scaled kernels. The scaling is ϵ^{-1}, reflecting that the mean number of particles per unit interval must converge.

Lemma 8. *For $\epsilon > 0$, let $X^{(\epsilon)}$ be a Pfaffian point process on $\epsilon\mathbb{Z}$ with kernel $\mathbf{K}^{(\epsilon)}$. Suppose that*

$$\sup_{\epsilon>0} \|\epsilon^{-1}\mathbf{K}^{(\epsilon)}\|_\infty = \sup_{\epsilon>0} \max_{i,j\in\{1,2\}} \sup_{x,y\in\epsilon\mathbb{Z}, x\le y} \epsilon^{-1}|\mathbf{K}_{ij}^{(\epsilon)}(x,y)| < \infty, \tag{4.1}$$

and

$$\lim_{\epsilon\downarrow 0} \epsilon^{-1}\mathbf{K}_{ij}^{(\epsilon)}(x_\epsilon, y_\epsilon) = \mathbf{K}_{ij}(x,y), \qquad \text{for } i,j \in \{1,2\}, \tag{4.2}$$

$$\text{when } (x_\epsilon, y_\epsilon) \to (x,y) \text{ with } x < y, \text{ or when } x_\epsilon = y_\epsilon \to y = x,$$

for some continuum kernel $\mathbf{K} : \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$. Then the point processes converge $X^{(\epsilon)} \to X$ in distribution as $\epsilon \downarrow 0$, on the space $\mathcal{M}_{\mathrm{LFP}}(\mathbb{R})$ equipped with the topology of vague convergence, and the limit X is a Pfaffian point process with kernel $\mathbf{K}$.

For each discrete model of chapter 3, the limiting kernel $\mathbf{K}(x,y)$ is discontinuous at $x = y$, leading to the careful point-wise convergence condition in the statement.

Proof of lemma 8. We follow the standard two-step proof for convergence in distribution: establish tightness and uniqueness of limits. The method hinges on the subsequence principle, asserting that convergence holds if and only if there is a limit such that any subsequence has a further subsequence, convergent to said limit. By Prohorov's theorem, tightness is equivalent to relative compactness in distribution, meaning that any subsequence of $(X^{(\epsilon)} : \epsilon > 0)$ contains a further convergent subsequence. It then remains, by the subsequence principle, to establish a unique limit for convergent sequences. A short introduction to the vague topology may be found at the end of section 2.2.2. We use Kallenberg [32] as the standard reference for probability theory.

Tightness follows from the fact that the first moments of $X^{(\epsilon)}$ may be uniformly bounded. Indeed to establish tightness of $(X^{(\epsilon)} : \epsilon > 0)$ as elements of $\mathcal{M}_{\mathrm{LFP}}(\mathbb{R})$, one must show that for any $\beta > 0$ there exists a compact set $U \subset \mathcal{M}_{\mathrm{LFP}}(\mathbb{R})$ such that $\mathbb{P}\left[X^{(\epsilon)} \notin U\right] \leq \beta$ for all $\epsilon > 0$. For $\gamma > 0$ define the set U_γ by

$$U_\gamma = \left\{ \mu \in \mathcal{M}_{\mathrm{LFP}}(\mathbb{R}) : \mu\left([-2^n, 2^n]\right) \leq \frac{4^n}{\gamma} \text{ for all } n \in \mathbb{N} \right\}.$$

Note that U_γ is relatively compact in the vague topology, since $\sup_{\mu \in U_\gamma} \mu f < \infty$ for all positive continuous $f : \mathbb{R} \to \mathbb{R}$ with compact support ([32], p. 564). The closure $\mathrm{cl}\,(U_\gamma) \subset \mathcal{M}_{\mathrm{LFP}}(\mathbb{R})$ is compact and the union bound and Markov inequality give

$$\begin{aligned}
\mathbb{P}\left[X^{(\epsilon)} \notin \mathrm{cl}\,(U_\gamma)\right] &\leq \mathbb{P}\left[X^{(\epsilon)} \notin U_\gamma\right] \\
&\leq \sum_{n=1}^{\infty} \mathbb{P}\left[X^{(\epsilon)}([-2^n, 2^n]) > \frac{4^n}{\gamma}\right] \\
&\leq \sum_{n=1}^{\infty} \frac{\gamma}{4^n} \mathbb{E}\left[X^{(\epsilon)}([-2^n, 2^n])\right].
\end{aligned}$$

We now bound the first moments uniformly in ϵ. Fix $a < b$, then writing in terms of the one-point intensity

$$\mathbb{E}\left[X^{(\epsilon)}([a,b])\right] = \sum_{x \in \epsilon\mathbb{Z} \cap [a,b]} \rho^{(1)}(x) = \sum_{x \in \epsilon\mathbb{Z}} \mathbf{K}_{12}^{(\epsilon)}(x,x) 1(x \in [a,b]).$$

Fix $M > 0$ such that $\sup_{\epsilon > 0} \|\epsilon^{-1}\mathbf{K}^{(\epsilon)}\|_\infty < M$. We may bound the first moment

$$\begin{aligned}
\mathbb{E}\left[X^{(\epsilon)}([a,b])\right] &= \int_{\mathbb{R}} \epsilon^{-1} \mathbf{K}_{12}^{(\epsilon)}\left(\epsilon\lfloor x\epsilon^{-1}\rfloor, \epsilon\lfloor x\epsilon^{-1}\rfloor\right) 1\left(\epsilon\lfloor x\epsilon^{-1}\rfloor \in [a,b]\right) \mathrm{d}x \\
&\leq M \int_{\mathbb{R}} 1\left(\epsilon\lfloor x\epsilon^{-1}\rfloor \in [a,b]\right) \mathrm{d}x \\
&\leq M(b - a + \epsilon).
\end{aligned}$$

Since we are interested in the limit $\epsilon \downarrow 0$ we may assume $\epsilon \leq 1$. Substituting in gives the following uniform bound

$$\mathbb{P}\left[X^{(\epsilon)} \notin \mathrm{cl}\,(U_\gamma)\right] \leq \gamma M \sum_{n=1}^{\infty} \frac{2^{n+1} + \epsilon}{4^n} \leq \gamma M \left(2 + \frac{1}{3}\right).$$

The right-hand side can be made arbitrarily small by taking γ sufficiently small and we have proved tightness.

Take an arbitrary subsequence of $X^{(\epsilon)}$ and consider by Prohorov's theorem ([32], p. 309) a further convergent subsequence, for convenience also denoted by $X^{(\epsilon)}$. The space $\mathcal{M}_{\mathrm{LFP}}(\mathbb{R})$ equipped with the topology of vague convergence is Polish ([32], p. 564) and hence so too is the set of probability measures on $\mathcal{M}_{\mathrm{LFP}}(\mathbb{R})$ equipped with the topology of weak convergence. In particular the latter space is closed, whereby the subsequence $X^{(\epsilon)}$ is convergent to a limit law on $\mathcal{M}_{\mathrm{LFP}}(\mathbb{R})$, which we denote by Y. It remains to show that Y is uniquely determined and has the same distribution as X. By proposition 13 the law of X is determined by virtue of it being a Pfaffian point process with bounded kernel $\mathbf{K}$, where boundedness follows from assumptions (4.1) and (4.2)

$$\sup_{x,y\in\mathbb{R}} \max_{i,j\in\{1,2\}} |\mathbf{K}_{ij}(x,y)| = \sup_{x,y\in\mathbb{R}} \lim_{\epsilon\downarrow 0} \max_{i,j\in\{1,2\}} \epsilon^{-1}|\mathbf{K}_{ij}^{(\epsilon)}(x,y)| \leq \lim_{\epsilon\downarrow 0} \|\epsilon^{-1}\mathbf{K}^{(\epsilon)}\|_\infty < \infty.$$

It therefore suffices to show that Y is Pfaffian with kernel $\mathbf{K}$. This is ultimately achieved by identifying the intensities, but we must first show that Y concentrates on the subset of simple measures $\mathcal{M}_0(\mathbb{R}) \subset \mathcal{M}_{\mathrm{LFP}}(\mathbb{R})$. This must be checked because $\mathcal{M}_0(\mathbb{R})$ is not a closed subset of $\mathcal{M}_{\mathrm{LFP}}(\mathbb{R})$. To this end, we repeat the above moment calculation for the second factorial moment. Using proposition 11 to introduce intensities

$$\mathbb{E}\left[\lfloor X^{(\epsilon)}([a,b])\rfloor_2\right] = \sum_{x,y\in\epsilon\mathbb{Z}\cap[a,b]} \rho^{(2)}(x,y) = \sum_{x,y\in\epsilon\mathbb{Z}\cap[a,b]} \mathrm{Pf}\left(\mathbf{K}^{(\epsilon)}\big((x,y)\big)\right).$$

Writing as an integral and moving the ϵ^{-1} factors onto the kernel (proposition 7)

$$\mathbb{E}\left[\lfloor X^{(\epsilon)}([a,b])\rfloor_2\right] = \int_{\mathbb{R}^2} \mathrm{Pf}\left(\epsilon^{-1}\mathbf{K}^{(\epsilon)}\Big(\big(\epsilon\lfloor x\epsilon^{-1}\rfloor, \epsilon\lfloor y\epsilon^{-1}\rfloor\big)\Big)\right) 1\left(\epsilon\lfloor x\epsilon^{-1}\rfloor, \epsilon\lfloor y\epsilon^{-1}\rfloor \in [a,b]\right) \mathrm{d}x\mathrm{d}y.$$

Finally, the kernel bound and (2.6) give

$$\mathbb{E}\left[\lfloor X^{(\epsilon)}([a,b])\rfloor_2\right] \leq 16M^2(b-a+\epsilon)^2. \tag{4.3}$$

We show that the bound also holds for the limit law

$$\mathbb{E}\left[\lfloor Y([a,b])\rfloor_2\right] \leq \mathbb{E}\left[\lfloor X^{(\epsilon)}([a,b])\rfloor_2\right].$$

Firstly, note that $X^{(\epsilon)} \to Y$ is equivalent to convergence in distribution of the random variables $X^{(\epsilon)}f \to Yf$ for all continuous $f : \mathbb{R} \to [0,\infty)$ with compact

support, where $X^{(\epsilon)}f = \int_{\mathbb{R}} f(x)X^{(\epsilon)}(\mathrm{d}x)$. Define functions $h : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R} \to [0, \infty)$ by $h(x) = x(x-1)$ and $f(x) = 1(x \in [a, b])$. For m, $n \in \mathbb{N}$ introduce continuous compactly supported approximations $h_m : \mathbb{R} \to \mathbb{R}$ and $f_n : \mathbb{R} \to [0, \infty)$, satisfying $h_m(x) \nearrow h(x)$ and $f_n(x) \nearrow f(x)$, by

$$h_m(x) = \begin{cases} h(x)1(x \in [-m, m]) & \text{if } x \in (-\infty, -m-1] \cup [-m, m] \cup [m+1, \infty), \\ \text{linear interpolation} & \text{if } x \in [-m-1, -m] \cup [m, m+1], \end{cases}$$

$$f_n(x) = \begin{cases} 1(x \in [a + 1/n, b - 1/n]) & \text{if } x \in [-\infty, a] \cup [a + 1/n, b - 1/n] \cup [b, \infty), \\ \text{linear interpolation} & \text{if } x \in [a, a + 1/n] \cup [b - 1/n, b]. \end{cases}$$

Noting that $h(x) \leq h(z)$ for $z \in \mathbb{N}$ and $0 \leq x \leq z$

$$\mathbb{E}\left[h_m\Big(X^{(\epsilon)}f_n\Big)\right] \leq \mathbb{E}\left[h\Big(X^{(\epsilon)}f_n\Big)\right] \leq \mathbb{E}\left[h\Big(X^{(\epsilon)}f\Big)\right] = \mathbb{E}\left[\lfloor X^{(\epsilon)}([a, b])\rfloor_2\right].$$

Taking limits in ϵ, the aforementioned convergence in distribution gives

$$\mathbb{E}\left[h_m(Yf_n)\right] \leq \mathbb{E}\left[\lfloor X^{(\epsilon)}([a, b])\rfloor_2\right].$$

It remains to take limits in n and m. The limit in n may be moved onto Yf_n by the dominated convergence theorem and continuity of h_m

$$\lim_{n\to\infty} \mathbb{E}\left[h_m(Yf_n)\right] = \mathbb{E}\left[h_m\Big(\lim_{n\to\infty} Yf_n\Big)\right].$$

The monotone convergence theorem guarantees that $\lim_{n\to\infty} Yf_n = Yf$ almost surely. Moreover, for the limit in m, the same theorem facilitates the exchange of limit and expectation

$$\lim_{m\to\infty} \mathbb{E}\left[h_m(Yf)\right] = \mathbb{E}\left[\lim_{m\to\infty} h_m(Yf)\right] = \mathbb{E}\left[h(Yf)\right] = \mathbb{E}\left[\lfloor Y([a, b])\rfloor_2\right].$$

Pulling everything together, we arrive at the claimed inequality

$$\mathbb{E}\left[\lfloor Y([a, b])\rfloor_2\right] = \lim_{m\to\infty}\lim_{n\to\infty} \mathbb{E}\left[h_m(Yf_n)\right] \leq \mathbb{E}\left[\lfloor X^{(\epsilon)}([a, b])\rfloor_2\right].$$

Combining with (4.3) and taking limits in ϵ, we obtain the uniform bound

$$\mathbb{E}\left[\lfloor Y([a, b])\rfloor_2\right] \leq 16M^2(b - a)^2.$$

Finally, the simplicity of Y follows from a routine covering argument

$$\begin{aligned}
\mathbb{P}\left[Y(\{x\}) \geq 2, \text{ for some } x \in [-L, L]\right] &\leq \sum_{i=-Lm}^{Lm} \mathbb{P}\left[Y\left(\left[\frac{i}{m}, \frac{i+1}{m}\right]\right) \geq 2\right] \\
&= \sum_{i=-Lm}^{Lm} \sum_{k=2}^{\infty} \mathbb{P}\left[Y\left(\left[\frac{i}{m}, \frac{i+1}{m}\right]\right) = k\right] \\
&\leq \sum_{i=-Lm}^{Lm} \sum_{k=0}^{\infty} (k^2 - k)\, \mathbb{P}\left[Y\left(\left[\frac{i}{m}, \frac{i+1}{m}\right]\right) = k\right] \\
&= \sum_{i=-Lm}^{Lm} \mathbb{E}\left[\left\lfloor Y\left(\left[\frac{i}{m}, \frac{i+1}{m}\right]\right)\right\rfloor_2\right] \\
&\leq 16M^2 \sum_{i=-Lm}^{Lm} m^{-2} \to 0 \quad \text{as } m \to \infty.
\end{aligned}$$

We now turn to deriving the intensities for Y. Fix mutually disjoint intervals $A_1, \ldots, A_k \subset \mathbb{R}$ and define the functional $\phi_k : \mathcal{M}_{\mathrm{LFP}}(\mathbb{R}) \to \mathbb{R}$ by $\phi_k(\mu) = \prod_{i=1}^{k} \mu(A_i)$. The aim is to take limits in the expectations $\mathbb{E}\left[\phi_k(X^{(\epsilon)})\right]$. Since the functional ϕ_k is neither continuous nor bounded, convergence in distribution does not allow us *a priori* to replace $X^{(\epsilon)}$ by Y. However, note that any measure in the discontinuity set of ϕ_k has a point mass at the boundary of some interval A_i. Moreover the first moment bound for $X^{(\epsilon)}$ passes to the limit

$$\mathbb{E}\left[Y([a, b])\right] \leq M(b - a),$$

so the discontinuity set is not charged by Y. (The bound can be proved as for $\mathbb{E}\left[\lfloor Y([a, b])\rfloor_2\right]$, replacing h by $h(x) = x$.) Hence, by the continuous mapping theorem ([32], p. 76), we have convergence in distribution of the random variables $\phi_k(X^{(\epsilon)}) \to \phi_k(Y)$. In order to conclude the desired convergence of means, it suffices to show that the family $\phi_k(X^{(\epsilon)})$ are uniformly integrable. Indeed Skorokhod's representation theorem guarantees existence of copies of the random variables, on some abstract probability space, converging almost surely ([32], p. 79). These copies are still uniformly integrable and Vitali's convergence theorem implies convergence in mean, and hence convergence of means ([46], p. 94). Since the mean is determined by the law, the result holds for the original random variables. We now check

uniform integrability of $\phi_k(X^{(\epsilon)})$, which follows from the kernel bound. For $L > 0$

$$\mathbb{E}\left[\phi_k(X^{(\epsilon)})1(\phi_k(X^{(\epsilon)}) \geq L)\right] \leq \frac{1}{L}\mathbb{E}\left[\phi_k(X^{(\epsilon)})^2\right] = \frac{1}{L}\mathbb{E}\left[\prod_{i=1}^{k} X^{(\epsilon)}(A_i)^2\right].$$

To incorporate intensities we write the expectation in terms of factorial moments

$$\begin{aligned}\mathbb{E}\left[\prod_{i=1}^{k} X^{(\epsilon)}(A_i)^2\right] &= \mathbb{E}\left[\prod_{i=1}^{k}\left(\lfloor X^{(\epsilon)}(A_i)\rfloor_2 + X^{(\epsilon)}(A_i)\right)\right] \\ &= \sum_{r_1=1}^{2}\cdots\sum_{r_k=1}^{2}\mathbb{E}\left[\prod_{i=1}^{k}\lfloor X^{(\epsilon)}(A_i)\rfloor_{r_i}\right].\end{aligned}$$

Suppose that $A_i = [a_i, b_i]$. Fix $r_1, \ldots, r_k$ and set $R = r_1 + \cdots + r_k$, then we develop the expectation by using proposition 11, moving constants onto the kernel and applying (2.6)

$$\begin{aligned}\mathbb{E}\left[\prod_{i=1}^{k}\lfloor X^{(\epsilon)}(A_i)\rfloor_{r_i}\right] &= \sum_{(x_1,\ldots,x_R)\in\prod_{i=1}^{k}A_i^{r_i}} \mathrm{Pf}\left(\mathbf{K}^{(\epsilon)}(x_i,x_j) : i,j \leq R\right) \\ &= \int_{\mathbb{R}^R}\mathrm{Pf}\left(\epsilon^{-1}\mathbf{K}^{(\epsilon)}\left(\epsilon\lfloor x_i\epsilon^{-1}\rfloor, \epsilon\lfloor x_j\epsilon^{-1}\rfloor\right) : i,j \leq R\right)\cdot \\ &\qquad \cdot 1\left((\epsilon\lfloor x_1\epsilon^{-1}\rfloor, \ldots, \epsilon\lfloor x_R\epsilon^{-1}\rfloor) \in \prod_{i=1}^{k}A_i^{r_i}\right)\mathrm{d}x_1\ldots\mathrm{d}x_R \\ &\leq (2M)^R R^R \prod_{i=1}^{k}(b_i - a_i + \epsilon)^{r_i}.\end{aligned}$$

The right-hand side can be bounded independently of $r_1, \ldots, r_k$ and uniformly in ϵ, for example by

$$C(k, M, \{A_i\}_{i=1}^k) = (2\max\{M,1\})^{2k}(2k)^{2k}\max_{1\leq i\leq k}\{(b_i - a_i + 1)^2\} < \infty.$$

Combining everything, we obtain the uniform bound

$$\mathbb{E}\left[\phi_k(X^{(\epsilon)})1(\phi_k(X^{(\epsilon)}) \geq L)\right] \leq \frac{C(k, M, \{A_i\}_{i=1}^k)2^k}{L},$$

which can be made arbitrarily small by varying L. Uniform integrability of $\phi_k(X^{(\epsilon)})$ is established.

We may now pass to the limit as $\epsilon \downarrow 0$ in the expectations

$$\mathbb{E}[\phi_k(X^{(\epsilon)})] \to \mathbb{E}\left[\phi_k(Y)\right].$$

On the other hand, the expectations are given in terms of intensities

$$\begin{aligned}
\mathbb{E}[\phi_k(X^{(\epsilon)})] &= \sum_{x_i \in \epsilon\mathbb{Z} \cap A_i} \rho^{(k)}(x_1, \ldots, x_k) \\
&= \sum_{x_i \in \epsilon\mathbb{Z}} \operatorname{Pf}\left(\mathbf{K}^{(\epsilon)}(x_i, x_j) : i, j \leq k\right) \prod_{i=1}^{k} 1\left(x_i \in A_i\right) \\
&= \int_{\mathbb{R}^k} \operatorname{Pf}\left(\epsilon^{-1}\mathbf{K}^{(\epsilon)}(\epsilon\lfloor x_i\epsilon^{-1}\rfloor, \epsilon\lfloor x_j\epsilon^{-1}\rfloor) : i, j \leq k\right) \cdot \\
&\qquad \cdot \textstyle\prod_{i=1}^{k} 1\left(\epsilon\lfloor x_i\epsilon^{-1}\rfloor \in A_i\right) \, \mathrm{d}x_1 \ldots \mathrm{d}x_k \\
&\to \int_{\prod_{i=1}^{k} A_i} \operatorname{Pf}\left(\mathbf{K}(x_i, x_j) : i, j \leq k\right) \mathrm{d}x_1 \ldots \mathrm{d}x_k.
\end{aligned}$$

The convergence of integrals follows from the assumptions. Indeed, (4.2) implies that the integrand converges pointwise and (4.1) facilitates the exchange of limit and integral by the dominated convergence theorem. Combining the two preceding displays, uniqueness of limits for real sequences gives

$$\mathbb{E}\left[\phi_k(Y)\right] = \int_{\prod_{i=1}^{k} A_i} \operatorname{Pf}\left(\mathbf{K}(x_i, x_j) : i, j \leq k\right) \mathrm{d}x_1 \ldots \mathrm{d}x_k.$$

This identifies the intensities of Y (see remark 2), proving it is a Pfaffian point process with kernel $\mathbf{K}$. This completes the proof. □

A similar result holds if the approximating sequences of points processes are defined on $\mathbb{R}$. In this case, there is another parameter λ in which limits are taken. For interacting particle systems the parameter is typically taken to be time, in order to study the large-time asymptotics $\lambda \to \infty$.

Lemma 9. *For $\lambda > 0$, let $X^{(\lambda)}$ be a Pfaffian point process on $\mathbb{R}$ with kernel $\mathbf{K}^{(\lambda)}$. Suppose that*

$$\sup_{\lambda > 1} \|\mathbf{K}^{(\lambda)}\|_\infty = \sup_{\lambda > 1} \max_{i,j \in \{1,2\}} \sup_{x,y \in \mathbb{R}} |\mathbf{K}_{ij}^{(\lambda)}(x, y)| < \infty, \tag{4.4}$$

and

$$\lim_{\lambda \to \infty} \mathbf{K}_{ij}^{(\lambda)}(x, y) = \mathbf{K}_{ij}(x, y), \qquad \text{for } i, j \in \{1, 2\}, \tag{4.5}$$

for some continuum kernel $\mathbf{K} : \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$. *Then the point processes converge* $X^{(\lambda)} \to X$ *in distribution as* $\lambda \to \infty$, *on the space* $\mathcal{M}_{\mathrm{LFP}}(\mathbb{R})$ *equipped with the topology of vague convergence, and the limit* X *is a Pfaffian point process with kernel* $\mathbf{K}$.

Proof of lemma 9. The proof translates directly from lemma 8, since the discrete point processes are already considered as elements of $\mathcal{M}_{\mathrm{LFP}}(\mathbb{R})$. The sums over $\epsilon\mathbb{Z}$ are replaced by integrals over $\mathbb{R}$. □

4.2 Limits of coalescing and annihilating random walks

The scaling theory is developed for the CARW model of section 3.2 in the homogeneous symmetric case.

We begin by proving in section 4.2.1 convergence of the scaled CARW point processes for symmetric rates and independent initial conditions. This gives a single maximal continuum initial condition and the limit point processes are determined by the particle rates. In section 4.2.2 we extend the results to independent initial conditions that scale with the discrete models, giving a family of Poisson limit initial conditions. We close the section by discussing some further models, in each case identifying the limiting continuum kernel.

4.2.1 Symmetric rates and independent initial conditions

The underling process is the CARW model of section 3.2, denoted $(X_t : t \geq 0)$, with rates $p_x = q_x = 1$ for $x \in \mathbb{Z}$ and interaction parameter $\theta \in [0, 1]$. We take the initial condition to be independent Bernoulli(λ) for some $\lambda \in (0, 1]$, noting that $\lambda = 0$ would correspond to the empty process. The model consists of coalescing and annihilating symmetric random walks and in particular the system is spatially homogeneous. This homogeneity simplifies the proof of convergence and enables explicit Gaussian expressions for the limit kernel. We now describe the scaled point processes.

Diffusive scaling theory for individual random walks is well understood and informs a suitable scaling for the associated interacting particle systems. For $\epsilon > 0$ the scaled point process $X^{(\epsilon)}$ on $\epsilon\mathbb{Z}$ is constructed by scaling diffusively

$$X^{(\epsilon)}(\mathrm{d}x) = X_{\epsilon^{-2}t}(\epsilon^{-1}\mathrm{d}x) \qquad \text{on } \epsilon\mathbb{Z}. \tag{4.6}$$

Note that one could carry t through to the subscript of $X^{(\epsilon)}$, however it is omitted to avoid confusion, since the convergence here is at a single (scaled) time and we do not claim existence of an underlying limiting stochastic process evolving in time. The aim is to apply lemma 8 to prove convergence in distribution of $X^{(\epsilon)}$. To prove convergence of Pfaffian point processes on $\epsilon\mathbb{Z}$ to a point process on $\mathbb{R}$, it suffices to check suitable convergence of the kernel entries, namely conditions (4.1) and (4.2). With this in mind, we take a closer look at the kernel $\mathbf{K}^{(\epsilon)}$ for $X^{(\epsilon)}$.

By theorem 1 and remark 6 the underlying point process X_t at time $t > 0$ is Pfaffian with kernel $\mathbf{K}(x, y)$, given, for $x < y$, by

$$\mathbf{K}(x,y) = \frac{-1}{1+\theta}\begin{pmatrix} K_t(x,y) & D_2^+ K_t(x,y) \\ D_1^+ K_t(x,y) & D_1^+ D_2^+ K_t(x,y) \end{pmatrix},$$

and $\mathbf{K}_{12}(x,x) = \frac{-1}{1+\theta} D_2^+ K_t(x,x)$, where $(K_t(x,y) : t \geq 0,\, x, y \in \mathbb{Z},\, x < y)$ is the unique bounded solution to following discrete heat equation on a wedge

$$\begin{cases} \partial_t K_t(x,y) &= (\Delta_x^{(1)} + \Delta_y^{(1)}) K_t(x,y) & \text{for } x < y,\, t > 0, \\ K_t(x,x) &= 1 & \text{for all } x,\, t > 0, \\ K_0(x,y) &= (1-(1+\theta)\lambda)^{y-x} & \text{for } x \leq y. \end{cases}$$

The one-particle generator, given by (3.7), is the central discrete Laplacian, defined by

$$\Delta^{(\epsilon)} f(x) = \epsilon^{-2} \left(f(x+\epsilon) + f(x-\epsilon) - 2f(x) \right), \tag{4.7}$$

where $f : \mathbb{Z} \to \mathbb{R}$ and a subscript indicates in which variable the operator acts. Consider now the scaled point process $X^{(\epsilon)}$. We define a scaled kernel function $K_t^{(\epsilon)}(x,y)$ by

$$K_t^{(\epsilon)}(x,y) = K_{\epsilon^{-2}t}(\epsilon^{-1}x, \epsilon^{-1}y) \qquad \text{for } x, y \in \epsilon\mathbb{Z}, \tag{4.8}$$

and introduce the following discrete right derivative approximations on $\epsilon\mathbb{Z}^2$

$$\begin{aligned} D_\epsilon^{(1,0)} f(x,y) &= \epsilon^{-1} \left(f(x+\epsilon, y) - f(x,y) \right), \\ D_\epsilon^{(0,1)} f(x,y) &= \epsilon^{-1} \left(f(x, y+\epsilon) - f(x,y) \right), \\ D_\epsilon^{(1,1)} f(x,y) &= \epsilon^{-2} \left(f(x+\epsilon, y+\epsilon) - f(x+\epsilon, y) - f(x, y+\epsilon) + f(x,y) \right), \end{aligned}$$

for $f : \epsilon\mathbb{Z}^2 \to \mathbb{R}$. Since diffusive scaling is just a relabelling of the underlying process, theorem 1 implies that $X^{(\epsilon)}$ is a Pfaffian point process on $\epsilon\mathbb{Z}$. The kernel $\tilde{\mathbf{K}}^{(\epsilon)}(x,y)$

may be expressed, for $x < y$, by

$$\tilde{\mathbf{K}}^{(\epsilon)}(x,y) = \frac{-1}{1+\theta}\begin{pmatrix} K_t^{(\epsilon)}(x,y) & \epsilon D_\epsilon^{(0,1)} K_t^{(\epsilon)}(x,y) \\ \epsilon D_\epsilon^{(1,0)} K_t^{(\epsilon)}(x,y) & \epsilon^2 D_\epsilon^{(1,1)} K_t^{(\epsilon)}(x,y) \end{pmatrix}, \tag{4.9}$$

and $\tilde{\mathbf{K}}_{12}^{(\epsilon)}(x,x) = \frac{-\epsilon}{1+\theta} D_\epsilon^{(0,1)} K_t^{(\epsilon)}(x,x)$. With lemma 8 in mind, we redistribute the factors of ϵ (proposition (12), part (1)) to give an equivalent kernel $\mathbf{K}^{(\epsilon)}(x,y)$ for $X^{(\epsilon)}$

$$\mathbf{K}^{(\epsilon)}(x,y) = \frac{-\epsilon}{1+\theta}\begin{pmatrix} K_t^{(\epsilon)}(x,y) & D_\epsilon^{(0,1)} K_t^{(\epsilon)}(x,y) \\ D_\epsilon^{(1,0)} K_t^{(\epsilon)}(x,y) & D_\epsilon^{(1,1)} K_t^{(\epsilon)}(x,y) \end{pmatrix}, \tag{4.10}$$

for $x < y$ and $\mathbf{K}_{12}^{(\epsilon)}(x,x) = \frac{-\epsilon}{1+\theta} D_\epsilon^{(0,1)} K_t^{(\epsilon)}(x,x)$. The function $(K_t^{(\epsilon)}(x,y) : t \geq 0,\, x,y \in \epsilon\mathbb{Z},\, x < y)$ is also characterised as the unique bounded solution to the discrete heat equation

$$\begin{cases} \partial_t K_t^{(\epsilon)}(x,y) &=& (\Delta_x^{(\epsilon)} + \Delta_y^{(\epsilon)}) K_t^{(\epsilon)}(x,y) & \text{for } x < y,\, t > 0, \\ K_t^{(\epsilon)}(x,x) &=& 1 & \text{for all } x,\, t > 0, \\ K_0^{(\epsilon)}(x,y) &=& (1-(1+\theta)\lambda)^{\epsilon^{-1}(y-x)} & \text{for } x \leq y, \end{cases} \tag{4.11}$$

where uniqueness follows from appendix A as for the $\mathbb{Z}$ analogue. We prove the following result.

Theorem 5. *Fix $t > 0$, $\theta \in [0,1]$ and $\lambda \in (0,1]$. Then the scaled symmetric CARW point process $X^{(\epsilon)}$ with interaction parameter θ and initial condition independent Bernoulli(λ) converges in distribution to X^c, the Pfaffian point process on $\mathbb{R}$ with kernel $\mathbf{K}^c(x,y)$, given, for $x < y$, by*

$$\mathbf{K}^c(x,y) = \frac{-1}{1+\theta}\begin{pmatrix} K_t^c(x,y) & \partial_2 K_t^c(x,y) \\ \partial_1 K_t^c(x,y) & \partial_1\partial_2 K_t^c(x,y) \end{pmatrix}, \tag{4.12}$$

and $\mathbf{K}_{12}^c(x,x) = \frac{-1}{1+\theta}\partial_2 K_t^c(x,x)$, where $(K_t^c(x,y) : t \geq 0,\, x,y \in \mathbb{R},\, x < y)$ is the unique bounded solution to the heat equation

$$\begin{cases} \partial_t K_t^c(x,y) &=& (\Delta_x + \Delta_y) K_t^c(x,y) & \textit{for } x < y,\, t > 0, \\ K_t^c(x,x) &=& 1 & \textit{for all } x,\, t > 0, \\ K_0^c(x,y) &=& 0 & \textit{for } x \leq y. \end{cases} \tag{4.13}$$

In particular, $K_t^c(x,y)$ may be written explicitly, for $x < y$ and $t \geq 0$, as

$$K_t^c(x,y) = \operatorname{erfc}\left(\frac{y-x}{2\sqrt{2t}}\right). \tag{4.14}$$

Remark 14. Scaling the jump rates to $p_x = q_x = a$ for $a > 0$ may be realised as a scaling $t \mapsto at$ in time. The one-particle generator for Brownian motion is $\frac{1}{2}\Delta$ and, as expected, computing the derivatives of $K_t^c(x, y)$ explicitly and setting $a = 1/2$, the kernel $\mathbf{K}^c$ coincides, at least when $\theta = 0$ or 1, with the kernel for interacting Brownian motions under a maximal entrance law [59]. In particular, for $\theta = 1$ the kernel $\mathbf{K}^c$ coincides with $\tilde{\mathbf{K}}_t^{\mathrm{ABM}}$ in (2.13) (after using proposition 12 to swap the order of entries). This kernel is equivalent to the Pfaffian kernel for the positions of real eigenvalues in the real Ginibre random matrix ensemble in the bulk limit as $N \to \infty$. For $\theta = 0$, the kernel $\mathbf{K}^c$ coincides with $\tilde{\mathbf{K}}_t^{\mathrm{CBM}}$ (after swapping the order of entries). See examples 4 and 6 for more details.

Remark 15. The parameter λ plays no role in the limit point process. For any $\lambda \in (0, 1]$ the expected number of particles for the scaled process at time zero in a bounded set is proportional to ϵ^{-1} and in the limit the particles become dense in the real line. In order to preserve an initial condition parameter in the limit, one should scale λ with ϵ, in the form $\lambda^{(\epsilon)}\epsilon^{-1} \to \mu > 0$. We develop this intuition in section 4.2.2.

Proof of theorem 5. The desired convergence follows from lemma 8 provided we can establish the boundedness and pointwise convergence conditions (4.1) and (4.2) for the kernels $\epsilon^{-1}\mathbf{K}^{(\epsilon)}$ (converging to $\mathbf{K}^c$), where $\epsilon^{-1}\mathbf{K}^{(\epsilon)}$ is given, for $x < y$, by

$$\epsilon^{-1}\mathbf{K}^{(\epsilon)}(x, y) = \frac{-1}{1+\theta}\begin{pmatrix} K_t^{(\epsilon)}(x, y) & D_\epsilon^{(0,1)}K_t^{(\epsilon)}(x, y) \\ D_\epsilon^{(1,0)}K_t^{(\epsilon)}(x, y) & D_\epsilon^{(1,1)}K_t^{(\epsilon)}(x, y) \end{pmatrix}, \tag{4.15}$$

and $\epsilon^{-1}\mathbf{K}_{12}^{(\epsilon)}(x, x) = \frac{-1}{1+\theta} D_\epsilon^{(0,1)}K_t^{(\epsilon)}(x, x)$. The outline is to prove uniform convergence of the kernel functions $K_t^{(\epsilon)}$ to K_t^c, along with the first and second derivatives, and show that this is enough to satisfy conditions (4.1) and (4.2). The first step is to recast the wedge heat equations (4.11) and (4.13) on $\epsilon\mathbb{Z}^2$ and $\mathbb{R}^2$. This facilitates standard PDE estimates for uniform convergence and explicit heat kernel solutions. To wit, forcing Dirichlet boundary conditions on the wedge PDEs (by considering auxiliary functions with 1 subtracted), then applying the method of images to write as a full-space PDE (by anti-symmetrising the initial conditions in the line $y = x$), we have, for $x < y$, the expressions

$$\begin{aligned} K_t^{(\epsilon)}(x, y) &= 1 + u_t^{(\epsilon)}(x, y), \\ K_t^c(x, y) &= 1 + u_t(x, y), \end{aligned} \tag{4.16}$$

where $(u_t^{(\epsilon)}(x,y) : t \geq 0,\, x,y \in \epsilon\mathbb{Z})$ solves the heat equation

$$\begin{cases} \partial_t u_t^{(\epsilon)}(x,y) &= (\Delta_x^{(\epsilon)} + \Delta_y^{(\epsilon)}) u_t^{(\epsilon)}(x,y) & \text{for } x,y \in \epsilon\mathbb{Z},\, t > 0, \\ u_0^{(\epsilon)}(x,y) &= \Big((1-(1+\theta)\lambda)^{\epsilon^{-1}|y-x|} - 1\Big)\operatorname{sgn}(y-x) & \text{for } x,y \in \mathbb{Z}, \end{cases}$$

and $(u_t(x,y) : t \geq 0,\, x,y \in \mathbb{R})$ solves the heat equation

$$\begin{cases} \partial_t u_t(x,y) &= (\Delta_x + \Delta_y) u_t(x,y) & \text{for } x,y \in \mathbb{R},\, t > 0, \\ u_0(x,y) &= \operatorname{sgn}(y-x) & \text{for } x,y \in \mathbb{R}. \end{cases} \tag{4.17}$$

The l_∞ norm on $\epsilon\mathbb{Z}^2$ is defined by $\|f\|_{l_\infty} = \sup_{x,y\in\epsilon\mathbb{Z}} |f(x,y)|$ for $f : \epsilon\mathbb{Z}^2 \to \mathbb{R}$ and the L_∞ norm on $\mathbb{R}^2$ by $\|f\|_{L_\infty} = \sup_{x,y\in\mathbb{R}^2} |f(x,y)|$ for $f : \mathbb{R}^2 \to \mathbb{R}$. We aim to prove the uniform convergence

$$\|D_\epsilon^\alpha u_t^{(\epsilon)} - D^\alpha u_t\|_{l_\infty} \to 0 \qquad \text{for } |\alpha| \leq 2, \text{ as } \epsilon \downarrow 0. \tag{4.18}$$

A sufficient condition for (4.18) in terms of convergence of the initial conditions is derived in appendix B. In particular, for bounded initial conditions $u_0^{(\epsilon)}$ and u_0 it suffices to check that

$$\|(u_0^{(\epsilon)} - P_\delta u_0) \star p_t^{(\epsilon)}\|_{l_\infty} \to 0 \qquad \text{as } \epsilon \downarrow 0, \tag{4.19}$$

where $\delta = \epsilon^k$ for some $k < 2$, P_t is the continuum semigroup for the heat equation, $p_t^{(\epsilon)} : \epsilon\mathbb{Z}^2 \to \mathbb{R}$ is the discrete heat kernel and the two-dimensional convolution on $\epsilon\mathbb{Z}^2$ is defined for $f, g : \epsilon\mathbb{Z}^2 \to \mathbb{R}$ by

$$(f \star g)(x,y) = \sum_{w_1,w_2 \in \epsilon\mathbb{Z}} f(x-w_1, y-w_2) g(w_1,w_2). \tag{4.20}$$

We return to prove that (4.19) holds for our particular initial conditions at the end. Note the first estimate of lemma 14 in appendix B gives uniqueness of u_t solving (4.17) within the class of continuously differentiable in time and twice continuously differentiable in space functions satisfying $\sup_{t\geq 0} \|u_t(x,y)\|_{L_\infty} < \infty$. By (4.16) this implies uniqueness for the limit wedge PDE (4.13).

We now show that conditions (4.1) and (4.2) follow from the uniform convergence (4.18). Considering first the boundedness condition, note that

$$\|D_\epsilon^\alpha u_t^{(\epsilon)}\|_{l_\infty} \leq \|D_\epsilon^\alpha u_t^{(\epsilon)} - D^\alpha u_t\|_{l_\infty} + \|D^\alpha u_t\|_{l_\infty}.$$

The first term on the right-hand side converges to zero by (4.18) and the second term on the right-hand side is independent of ϵ and bounded, for example by the first estimate in lemma 14. By (4.16) this gives a uniform bound on

$$\max_{i,j\in\{1,2\}} \epsilon^{-1} \|D_\epsilon^\alpha \mathbf{K}_{ij}^{(\epsilon)}(x,y)\|_{l_\infty},$$

and condition (4.1) is satisfied. Suppose now that x_ϵ, $y_\epsilon \in \epsilon\mathbb{Z}$ and x, $y \in \mathbb{R}$ satisfy $x_\epsilon \to x$ and $y_\epsilon \to y$ or $x_\epsilon = y_\epsilon \to y = x$. Note that (4.18) implies $\left|D_\epsilon^\alpha u_t^{(\epsilon)}(x_\epsilon, y_\epsilon) - D^\alpha u_t(x_\epsilon, y_\epsilon)\right| \to 0$ and continuity of $D^\alpha u_t$ gives $\left|D^\alpha u_t(x_\epsilon, y_\epsilon) - D^\alpha u_t(x, y)\right| \to 0$. By the triangle inequality and (4.16) we find

$$\lim_{\epsilon\downarrow 0} \epsilon^{-1} \mathbf{K}_{ij}^{(\epsilon)}(x_\epsilon, y_\epsilon) = \mathbf{K}_{ij}(x,y), \qquad \text{for } i,j \in \{1,2\},$$

which is condition (4.2).

It remains to show that the limit (4.19) holds for the initial conditions

$$\begin{aligned} u_0^{(\epsilon)}(x,y) &= \left((1-(1+\theta)\lambda)^{\epsilon^{-1}|y-x|} - 1\right) \operatorname{sgn}(y-x), \\ u_0(x,y) &= \operatorname{sgn}(y-x), \end{aligned}$$

which we note are bounded in $[-1,1]$. Developing the convolution formula for the solution to the heat equation (4.17) gives the following expression

$$u_t(x,y) = \iint_{\mathbb{R}^2} \frac{1}{4\pi t} e^{-\frac{(x-w_1)^2}{4t}} e^{-\frac{(y-w_2)^2}{4t}} \operatorname{sgn}(w_1 - w_2)\, \mathrm{d}w_1 \mathrm{d}w_2 = \operatorname{erf}\left(\frac{x-y}{2\sqrt{2t}}\right). \tag{4.21}$$

The claimed solution to the wedge PDE (4.13) is recovered from (4.16)

$$K_t^c(x,y) = 1 + u_t(x,y) = \operatorname{erfc}\left(\frac{y-x}{2\sqrt{2t}}\right).$$

For now we exclude the case $\theta = \lambda = 1$, returning to it at the end. Condition (4.19) demands control of

$$\|(u_0^{(\epsilon)} - u_\delta) \star p_t^{(\epsilon)}\|_{l_\infty} = \sup_{x,y\in\epsilon\mathbb{Z}} \left| \sum_{w_1,w_2\in\epsilon\mathbb{Z}} (u_0^{(\epsilon)} - u_\delta)(x-w_1, y-w_2)\, p_t^{(\epsilon)}(w_1,w_2) \right|,$$

where $\delta = \epsilon^k$ for some $k < 2$ and $p_t^{(\epsilon)}$ is the discrete heat kernel on $\epsilon\mathbb{Z}^2$, namely

$$\begin{cases} \partial_t p_t^{(\epsilon)}(x,y) &= (\Delta_x^{(\epsilon)} + \Delta_y^{(\epsilon)}) p_t^{(\epsilon)}(x,y) \quad \text{for } x,y \in \epsilon\mathbb{Z},\, t > 0, \\ p_0^{(\epsilon)}(x,y) &= 1(x=0)1(y=0) \qquad\qquad \text{for } x,y \in \epsilon\mathbb{Z}. \end{cases}$$

Substituting in for $u_0^{(\epsilon)}$ and u_δ gives

$$|u_0^{(\epsilon)} - u_\delta|(x,y) \le |1-(1+\theta)\lambda|^{\epsilon^{-1}|y-x|} + \operatorname{erfc}\left(\frac{|y-x|}{2\sqrt{2\delta}}\right), \tag{4.22}$$

for x, $y \in \epsilon\mathbb{Z}$. The outline is to split $\|(u_0^{(\epsilon)} - u_\delta) \star p_t^{(\epsilon)}\|_{l_\infty}$ into two parts and show that each vanishes as $\epsilon \downarrow 0$. For $\gamma > 0$, we introduce complimentary sets

$$A_1 = \{w_1, w_2 \in \mathbb{R} : |w_2 - w_1| \le \epsilon^\gamma\}, \quad A_2 = \{w_1, w_2 \in \mathbb{R} : |w_2 - w_1| > \epsilon^\gamma\},$$

and consider

$$\|(u_0^{(\epsilon)} - u_\delta) \star p_t^{(\epsilon)}\|_{l_\infty} \le \||u_0^{(\epsilon)} - u_\delta| 1(A_1) \star p_t^{(\epsilon)}\|_{l_\infty} + \||u_0^{(\epsilon)} - u_\delta| 1(A_2) \star p_t^{(\epsilon)}\|_{l_\infty}. \tag{4.23}$$

The intuition is that $|u_0^{(\epsilon)} - u_\delta|$ is small on A_2 and bounded on the thin strip A_1. Noting that $\|u_0^{(\epsilon)} - u_\delta\|_{l_\infty} \le 2$

$$\||u_0^{(\epsilon)} - u_\delta| 1(A_1) \star p_t^{(\epsilon)}\|_{l_\infty} \le 2 \sup_{x,y \in \epsilon\mathbb{Z}} \left(\sum_{w_1, w_2 \in \epsilon\mathbb{Z}} 1\left((x-w_1, y-w_2) \in A_1\right) p_t^{(\epsilon)}(w_1, w_2) \right).$$

The sum has a probabilistic interpretation, since $p_t^{(\epsilon)}$ is the transition function for a continuous-time random walk $\left(Z_t^{(\epsilon)} : t \ge 0\right)$ on $\epsilon\mathbb{Z}^2$. The components of $\left(Z_t^{(\epsilon)} : t \ge 0\right)$ are independent symmetric random walks on $\epsilon\mathbb{Z}$ with jump rate ϵ^{-2}, and $Z_0^{(\epsilon)} = (0,0)$. For fixed $x, y \in \epsilon\mathbb{Z}$

$$\sum_{w_1, w_2 \in \epsilon\mathbb{Z}} 1\left((x-w_1, y-w_2) \in A_1\right) p_t^{(\epsilon)}(w_1, w_2) = \mathbb{P}\left[Z_t^{(\epsilon)} \in (x,y) + A_1\right],$$

where $(x,y) + A_1 = \{w_1, w_2 \in \mathbb{R} : |w_2 - w_1 - (y-x)| \le \epsilon^\gamma\}$. The probability may be expressed in terms of the one-dimensional process $\left(Y_t^{(\epsilon)} : t \ge 0\right)$, the difference between the components of $Z_t^{(\epsilon)}$, which evolves as a symmetric random walk on $\epsilon\mathbb{Z}$ with jump rates $2\epsilon^{-2}$ and $Y_0^{(\epsilon)} = 0$

$$\mathbb{P}\left[Z_t^{(\epsilon)} \in (x,y) + A_1\right] = \mathbb{P}\left[\left|Y_t^{(\epsilon)} - (y-x)\right| \le \epsilon^\gamma\right].$$

This probability is maximised by taking $x = y$ and we arrive at the bound

$$\| |u_0^{(\epsilon)} - u_\delta| 1(A_1) \star p_t^{(\epsilon)} \|_{l_\infty} \leq \mathbb{P}\left[\left| Y_t^{(\epsilon)} \right| \leq \epsilon^\gamma \right].$$

The following lemma finishes the proof that the A_1 contribution vanishes as $\epsilon \downarrow 0$.

Lemma 10. *In the standing notation*

$$\lim_{\epsilon \downarrow 0} \mathbb{P}\left[\left| Y_t^{(\epsilon)} \right| \leq \epsilon^\gamma \right] = 0.$$

Proof of lemma 10. The proof is an exercise in applying Donsker's theorem to compare with Brownian probabilities. We show that for $\beta > 0$ there exists $\epsilon_0 > 0$ such that $\mathbb{P}\left[\left| Y_t^{(\epsilon)} \right| \leq \epsilon^\gamma \right] < \beta$ for $\epsilon < \epsilon_0$. By Donsker's theorem, the random walk $\left(Y_t^{(\epsilon)} : t \geq 0 \right)$ converges in distribution to the Brownian motion $\left(B_t : t \geq 0 \right)$ with diffusion coefficient 2. The event depends on ϵ and in order to compare with Brownian probabilities we consider an arbitrary strip width $L > 0$

$$\mathbb{P}\left[|B_t| \leq L \right] = \int_{-L}^{L} \frac{1}{2\sqrt{\pi t}} e^{-\frac{w^2}{4t}} \, \mathrm{d}w \leq \frac{CL}{\sqrt{t}}.$$

Let L_0 be such that $\mathbb{P}\left[|B_t| \leq L \right] < \beta/2$ for $L < L_0$. For sufficiently small ϵ, we can pick $L \in (\epsilon^\gamma, L_0)$ then

$$\mathbb{P}\left[\left| Y_t^{(\epsilon)} \right| \leq \epsilon^\gamma \right] \leq \mathbb{P}\left[\left| Y_t^{(\epsilon)} \right| \leq L \right].$$

The discontinuity set of the function $w \mapsto 1(|w| \leq L)$ has Lebesgue measure zero and is not charged by the law of B_t. Consequently, the probabilities converge and there exists $\epsilon_0 > 0$ such that for $\epsilon < \epsilon_0$

$$\left| \mathbb{P}\left[\left| Y_t^{(\epsilon)} \right| \leq L \right] - \mathbb{P}\left[|B_t| \leq L \right] \right| < \frac{\beta}{2}.$$

Summing with $\mathbb{P}\left[|B_t| \leq L \right]$ and applying the triangle inequality gives the desired bound. □

Turning to the second term of (4.23)

$$\| |u_0^{(\epsilon)} - u_\delta| 1(A_2) \star p_t^{(\epsilon)} \|_{l_\infty} \leq \| |u_0^{(\epsilon)} - u_\delta| 1(A_2) \|_{l_\infty} \| 1 \star p_t^{(\epsilon)} \|_{l_\infty}.$$

Note that $1 \star p_t^{(\epsilon)} = 1$, then substituting in (4.22) we arrive at the bound

$$\begin{aligned}\||u_0^{(\epsilon)} - u_\delta|1(A_2) \star p_t^{(\epsilon)}\|_{l_\infty} &\leq \sup_{x,y \in A_2} \left(|1-(1+\theta)\lambda|^{\epsilon^{-1}|y-x|} + \operatorname{erfc}\left(\frac{|x-y|}{2\sqrt{2\delta}}\right)\right) \\ &\leq |1-(1+\theta)\lambda|^{\epsilon^{\gamma-1}} + \operatorname{erfc}\left(\frac{\epsilon^\gamma}{2\sqrt{2\delta}}\right).\end{aligned}$$

Picking $\delta = \epsilon$ and $\gamma = 1/3$, for example, the right-hand side converges to zero as $\epsilon \downarrow 0$, and we have established condition (4.19).

We now turn to the case $\theta = \lambda = 1$. Note that the above method no longer works since $1-(1+\theta)\lambda = -1$ so the last bound does not converge to zero. The problem is that bounding $u_0^{(\epsilon)} - u_\delta$ by $|u_0^{(\epsilon)} - u_\delta|$ is too crude, as it discards cancellations. Note that in this case

$$(u_0^{(\epsilon)} - u_\delta)(x,y) = (-1)^{\epsilon^{-1}|y-x|} \operatorname{sgn}(y-x) + \operatorname{erfc}\left(\frac{|y-x|}{2\sqrt{2\delta}}\right) \operatorname{sgn}(x-y).$$

The second term does not depend on θ or λ, so may be handled as above. The remaining term of $\|(u_0^{(\epsilon)} - u_\delta) \star p_t^{(\epsilon)}\|_{l_\infty}$ to control is

$$\sup_{x,y \in \epsilon\mathbb{Z}} \left| \sum_{w_1,w_2 \in \epsilon\mathbb{Z}} (-1)^{\epsilon^{-1}|w_2-w_1|} \operatorname{sgn}(w_2-w_1) p_t^{(\epsilon)}(x-w_1, y-w_2) \right|. \tag{4.24}$$

Consider the function $(-1)^{\epsilon^{-1}|w_2-w_1|} \operatorname{sgn}(w_2-w_1)$ on $\epsilon\mathbb{Z}^2$. Note that $(-1)^{\epsilon^{-1}|w_2-w_1|}$ is an assignation of alternating ± 1 values. The factor $\operatorname{sgn}(w_2-w_1)$ turns the values on the diagonal set $\{w_2 = w_1\}$ to zero and changes the sign of those below the diagonal. Overall the graph has a diagonal of zeroes dividing two regions of alternating signs. An adjacent pair of alternating signs contributes a discrete derivative of $p_t^{(\epsilon)}$ to the sum, scaled by a factor of ϵ. Which argument the derivative acts on depends on how the adjacent pairs are grouped. For definiteness we pair terms in the w_1 plane, giving derivatives in w_1. For fixed x, $y \in \epsilon\mathbb{Z}$, the sum in (4.24) is bounded by pairing off the ± 1 terms as discrete derivatives and summing their absolute values. Taking the full sum on $\epsilon\mathbb{Z}^2$ gives an upper bound of $\epsilon \|D_\epsilon^{(1,0)} p_t^{(\epsilon)}\|_{l_1}$. The l_1 norm may be bounded by lemma 15 of appendix B, giving

$$\sup_{x,y \in \epsilon\mathbb{Z}} \left| \sum_{w_1,w_2 \in \epsilon\mathbb{Z}} (-1)^{\epsilon^{-1}|w_2-w_1|} \operatorname{sgn}(w_2-w_1) p_t^{(\epsilon)}(x-w_1, y-w_2) \right| \leq C\epsilon t^{-1/2},$$

for some $C > 0$ and for sufficiently small ϵ, namely $t \geq \epsilon^2$. The right-hand side converges to zero as $\epsilon \downarrow 0$, completing the proof in the case $\theta = \lambda = 1$. □

Remark 16. Alternatively, the multiplicative factor $1/(1+\theta)$ may be deduced from the case $\theta = 0$ by a thinning argument, a generalisation of the result for discrete models, see remarks 3 and 6.

4.2.2 Symmetric rates and scaled independent initial conditions

We generalise section 4.2.1 to ϵ-dependent initial conditions. Consider the scaled symmetric CARW point process $X^{(\epsilon)}$ on $\epsilon\mathbb{Z}$ with rates $p_x = q_x = 1$, interaction parameter $\theta \in [0, 1]$ and initial conditions that are Bernoulli(λ_ϵ) for $\lambda_\epsilon \in (0, 1]$. We assume that $\lambda_\epsilon \to \lambda_0 \in [0, 1]$ as $\epsilon \downarrow 0$. Inspecting the proof of theorem 5, the limit of the scaled point processes is determined by the convergence of the discrete initial conditions and in particular convergence of $(1 - (1 + \theta)\lambda_\epsilon)^{\epsilon^{-1}}$. The limit is given by

$$\lim_{\epsilon \downarrow 0} (1 - (1 + \theta)\lambda_\epsilon)^{\epsilon^{-1}} = e^{-(1+\theta)\mu}, \quad \text{where} \quad \mu = \lim_{\epsilon \downarrow 0} \epsilon^{-1}\lambda_\epsilon.$$

The limit point process is determined by the value of $\mu \in [0, \infty]$, which is the asymptotic density of particles in the initial condition. The case $\mu = 0$ corresponds to λ_ϵ converging to zero too fast for the process have any chance of propagating. The degenerate empty point process is obtained in the limit. The case $\mu = \infty$ corresponds to λ_ϵ converging to zero slowly or $\lambda_\epsilon \to \lambda_0 \in (0, 1]$, and we are back in the setting of theorem 5. Finally, $\mu \in (0, \infty)$ represents an intermediate scale, in which the mean density of particles is preserved in the limit, and a new family of point processes emerges. Underlying this is convergence of the independent Bernoulli point process to a Poisson process. We collect everything together in a single result, which generalises theorem 5.

Theorem 6. *Fix $t > 0$, $\theta \in [0, 1]$ and $\lambda_\epsilon \in [0, 1]$ for $\epsilon > 0$. Then the scaled symmetric CARW point process $X^{(\epsilon)}$ with interaction parameter θ and initial condition independent Bernoulli(λ_ϵ) converges in distribution to X^c, the Pfaffian point process on $\mathbb{R}$ with kernel $\mathbf{K}^c(x, y)$ of the form (4.12), where $(K_t^c(x, y) : t \geq 0,\ x, y \in \mathbb{R},\ x < y)$ is the unique bounded solution to the heat equation*

$$\begin{cases} \partial_t K_t^c(x,y) &= (\Delta_x + \Delta_y) K_t^c(x,y) & \text{for } x < y,\ t > 0, \\ K_t^c(x,x) &= 1 & \text{for all } x,\ t > 0, \\ K_0^c(x,y) &= e^{-(1+\theta)\mu(y-x)} & \text{for } x \leq y, \end{cases} \tag{4.25}$$

where $\mu = \lim_{\epsilon \downarrow 0} \epsilon^{-1}\lambda_\epsilon \in [0, \infty]$. In particular, $K_t^c(x, y)$ may be written explicitly,

for $x < y$ and $t \geq 0$, as

$$K_t^c(x,y) = \operatorname{erfc}\left(\frac{y-x}{2\sqrt{2t}}\right) + F_{\mu_\theta,t}(y-x) - F_{\mu_\theta,t}(x-y), \tag{4.26}$$

where $\mu_\theta = (1+\theta)\mu$ and $F_{\mu_\theta,t} : \mathbb{R} \to [0,\infty)$ is defined by

$$F_{\mu_\theta,t}(z) = \frac{e^{2\mu_\theta^2 t}}{2} e^{-\mu_\theta z} \operatorname{erfc}\left(\frac{4\mu_\theta t - z}{2\sqrt{2t}}\right). \tag{4.27}$$

In view of example 2 (redistributing constants in (2.7)) and the thinning result proposition 15, the point process for $t = 0$ is a rate μ Poisson process. By the comments after theorem 5, the case $\mu = \infty$ corresponds to the one-dimensional marginals of a system of interacting Brownian motions under a maximal entrance law (at least in the case of $\theta = 0$ or 1). Thus, it is natural to conjecture that the case of finite μ corresponds to the one-dimensional marginals of interacting Brownian motions started from rate μ Poisson initial conditions.

Remark 17. Since the result hinges on convergence of $(1-(1+\theta)\lambda_\epsilon)^{\epsilon^{-1}}$ we could just as well scale the interaction parameter θ with ϵ. Suppose $X^{(\epsilon)}$ is as in theorem 6 but with interaction parameter $\theta_\epsilon \in [0,1]$, then $X^{(\epsilon)}$ converges to X^c with kernel (4.12), characterised by (4.25) with θ replaced by θ_0 where $\theta_\epsilon \to \theta_0 \in [0,1]$. That is, the limit point process depends on the limit interaction parameter. The fact that the initial condition involves $1+\theta_\epsilon$ precludes any interesting interplay with the Bernoulli rate λ_ϵ.

Proof of theorem 6. The only difference to theorem 5 is the initial conditions, which remain bounded in $[-1,1]$. Inspecting the proof, it suffices to show that $|u_0^{(\epsilon)} - u_\delta|(x,y)$ is uniformly bounded on $\epsilon\mathbb{Z}^2$ and its supremum on $|y-x| > \epsilon^\gamma$ converges to zero as $\epsilon \downarrow 0$ for some $\gamma > 0$ and $\delta = \epsilon^k$ with $k < 2$.

We begin with the maximal case $\mu = \infty$. The corresponding kernel function (4.26) is given by (4.14) and the result follows by replacing λ in the proof of theorem 5 by λ_ϵ. Indeed note that $|1-(1+\theta)\lambda_\epsilon|^{\epsilon^{-1}|y-x|} \leq 1$ and, provided we are not in the case $\theta = \lambda_0 = 1$, $|1-(1+\theta)\lambda_\epsilon|^{\epsilon^{\gamma-1}} \to 0$ for $\gamma < 1$ as $\epsilon \downarrow 0$. For $\theta = \lambda_0 = 1$ we must consider separately an oscillating term of $\|(u_0^{(\epsilon)} - u_\delta) \star p_t^{(\epsilon)}\|_{l_\infty}$, namely the analogue of (4.24)

$$\sup_{x,y\in\epsilon\mathbb{Z}} \left| \sum_{w_1,w_2\in\epsilon\mathbb{Z}} (1-2\lambda_\epsilon)^{\epsilon^{-1}|w_2-w_1|} \operatorname{sgn}(w_2-w_1) p_t^{(\epsilon)}(x-w_1, y-w_2) \right|. \tag{4.28}$$

We split $1-2\lambda_\epsilon$ as $(-1)(2\lambda_\epsilon-1)$ and note that since $\lambda_\epsilon \to \lambda_0 = 1$ we may assume $2\lambda_\epsilon - 1 > 0$. The terms in the sum have the same signs as (4.24) and we group adjacent pairs as before. In this way, we bound by a sum of terms $\epsilon \left| D_\epsilon^{(1,0)} f(w_1, w_2) \right|$ where $f(w_1, w_2) = (2\lambda_\epsilon - 1)^{\epsilon^{-1}|w_2 - w_1|} p_t^{(\epsilon)}(x - w_1, y - w_2)$. Applying a discrete product rule and the triangle inequality, $\epsilon \left| D_\epsilon^{(1,0)} f(w_1, w_2) \right|$ is bounded by

$$
\begin{aligned}
&(2\lambda_\epsilon - 1)^{\epsilon^{-1}|w_2 - w_1|} \left| p_t^{(\epsilon)}(x - (w_1 + \epsilon), y - w_2) - p_t^{(\epsilon)}(x - w_1, y - w_2) \right| \\
&\quad + \left| (2\lambda_\epsilon - 1)^{\epsilon^{-1}|w_2 - (w_1+\epsilon)|} - (2\lambda_\epsilon - 1)^{\epsilon^{-1}|w_2 - w_1|} \right| \left| p_t^{(\epsilon)}(x - (w_1 + \epsilon), y - w_2) \right|.
\end{aligned}
$$

Note that $(2\lambda_\epsilon - 1)^{\epsilon^{-1}|w_2 - w_1|} \leq 1$, since $\epsilon^{-1}|w_2 - w_1| \in \mathbb{N}$, and

$$
\begin{aligned}
&\left| (2\lambda_\epsilon - 1)^{\epsilon^{-1}|w_2 - (w_1+\epsilon)|} - (2\lambda_\epsilon - 1)^{\epsilon^{-1}|w_2 - w_1|} \right| \\
&\qquad = \begin{cases} (2\lambda_\epsilon - 1)^{\epsilon^{-1}(w_1 - w_2 - \epsilon)} \left| 1 - (2\lambda_\epsilon - 1)^1 \right| & \text{if } w_2 > w_1, \\ (2\lambda_\epsilon - 1)^{\epsilon^{-1}(w_1 - w_2)} \left| (2\lambda_\epsilon - 1)^1 - 1 \right| & \text{if } w_2 \leq w_1. \end{cases}
\end{aligned}
$$

All together we find

$$
\begin{aligned}
\epsilon \left| D_\epsilon^{(1,0)} f(w_1, w_2) \right| \leq &\left| p_t^{(\epsilon)}(x - (w_1 + \epsilon), y - w_2) - p_t^{(\epsilon)}(x - w_1, y - w_2) \right| \\
&+ 2(1 - \lambda_\epsilon) \left| p_t^{(\epsilon)}(x - (w_1 + \epsilon), y - w_2) \right|.
\end{aligned}
$$

Taking the full sum in $\epsilon\mathbb{Z}^2$ gives the bound $\epsilon \| D_\epsilon^{(1,0)} p_t^{(\epsilon)} \|_{l_1} + 2(1 - \lambda_\epsilon) \| p_t^{(\epsilon)} \|_{l_1}$ for (4.28). Bounding the l_1 norms by lemma 15 of appendix B gives

$$
\epsilon \| D_\epsilon^{(1,0)} p_t^{(\epsilon)} \|_{l_1} + 2(1 - \lambda_\epsilon) \| p_t^{(\epsilon)} \|_{l_1} \leq C(\epsilon t^{-1/2} + 2(1 - \lambda_\epsilon)),
$$

for some $C > 0$ and for sufficiently small ϵ, namely $t \geq \epsilon^2$. The right-hand side converges to zero as $\epsilon \downarrow 0$ since $\lambda_\epsilon \to \lambda_0 = 1$. This completes the proof for $\mu = \infty$.

Consider now the case $\mu \in (0, \infty)$. The characterising heat equation on $\mathbb{R}^2$ is

$$
\begin{cases}
\partial_t u_t(x, y) &= (\Delta_x + \Delta_y) u_t(x, y) & \text{for } x, y \in \mathbb{R}^2,\ t > 0, \\
u_0(x, y) &= (e^{-(1+\theta)\mu|y-x|} - 1) \operatorname{sgn}(y - x) & \text{for } x, y \in \mathbb{R}^2.
\end{cases}
$$

By linearity we can solve the PDE by splitting the initial conditions. From theorem 5 we already have the explicit solution for the initial condition $\operatorname{sgn}(x-y)$. This solution is generalised to the initial condition $e^{-(1+\theta)\mu|y-x|} \operatorname{sgn}(y - x)$ by developing the heat

kernel convolution, namely

$$\iint_{\mathbb{R}^2} \frac{1}{4\pi t} e^{-\frac{(x-w_1)^2}{4t}} e^{-\frac{(y-w_2)^2}{4t}} e^{-\mu_\theta |w_2-w_1|} \operatorname{sgn}(w_2-w_1)\, \mathrm{d}w_1 \mathrm{d}w_2 \\ = \operatorname{sgn}(y-x)\big(F_{\mu_\theta,t}\big(|y-x|\big) - F_{\mu_\theta,t}\big(-|y-x|\big)\big), \quad (4.29)$$

where $\mu_\theta = (1+\theta)\mu$ and $F_{\mu_\theta,t}(z)$ is given by (4.27). The claimed expression (4.26) for $K_t^c(x,y)$ follows by summing the two solutions and from (4.16). All together, substituting in for $u_0^{(\epsilon)}$ and u_δ

$$|u_0^{(\epsilon)} - u_\delta|(x,y) \le \left| (1-(1+\theta)\lambda_\epsilon)^{\epsilon^{-1}|y-x|} - \frac{e^{2\mu_\theta^2\delta}}{2} e^{-\mu_\theta|y-x|} \operatorname{erfc}\left(\frac{4\mu_\theta\delta - |y-x|}{2\sqrt{2\delta}}\right)\right| \\ + \frac{e^{2\mu_\theta^2\delta}}{2} e^{\mu_\theta|y-x|} \operatorname{erfc}\left(\frac{4\mu_\theta\delta + |y-x|}{2\sqrt{2\delta}}\right) + \operatorname{erfc}\left(\frac{|y-x|}{2\sqrt{2\delta}}\right).$$

We derive a uniform bound on $|u_0^{(\epsilon)} - u_\delta|(x,y)$. Using the bound $\operatorname{erfc}(z) \le (\sqrt{\pi}z)^{-1}e^{-z^2}$ for $z>0$, note that

$$e^{\mu_\theta|y-x|} \operatorname{erfc}\left(\frac{4\mu_\theta\delta + |y-x|}{2\sqrt{2\delta}}\right) \le \frac{2\sqrt{2\delta}}{\sqrt{\pi}|y-x|} e^{|y-x|\left(\mu_\theta - (8\delta)^{-1}|y-x|\right)} \le \frac{2\sqrt{2\delta}}{\sqrt{\pi}L},$$

where the last equality holds for $|y-x| \ge L$ (with sufficiently small $\delta \le (8\mu_\theta)^{-1}L$). Bounding instead the error function by 1 gives the bound e^{μ_θ} for $|y-x|<1$. Combining with the last display for $L=1$, we arrive at the uniform bound

$$\|u_0^{(\epsilon)} - u_\delta\|_{l_\infty} \le 2 + e^{2\mu_\theta^2\delta}\left(1 + \sqrt{\frac{2\delta}{\pi}} + \frac{e^{\mu_\theta}}{2}\right) \le 2 + e^{2\mu_\theta^2}\left(1 + \sqrt{\frac{2}{\pi}} + \frac{e^{\mu_\theta}}{2}\right),$$

where the last inequality holds for sufficiently small $\delta < 1$. When $|y-x| > \epsilon^\gamma$, $|u_0^{(\epsilon)} - u_\delta|(x,y)$ is bounded as follows

$$|u_0^{(\epsilon)} - u_\delta|(x,y) \le \left| e^{-\mu_\theta|y-x|} - \frac{e^{2\mu_\theta^2\delta}}{2} e^{-\mu_\theta|y-x|} \operatorname{erfc}\left(\frac{4\mu_\theta\delta - |y-x|}{2\sqrt{2\delta}}\right)\right| \\ + \left|(1-(1+\theta)\lambda_\epsilon)^{\epsilon^{-1}|y-x|} - e^{-\mu_\theta|y-x|}\right| + e^{2\mu_\theta^2\delta}\frac{\sqrt{2\delta}}{\sqrt{\pi}\epsilon^\gamma} + \operatorname{erfc}\left(\frac{\epsilon^\gamma}{2\sqrt{2\delta}}\right).$$

Taking $\delta = \epsilon$ and $\gamma = 1/3$, the right-hand side converges to zero as $\epsilon \downarrow 0$. Indeed the convergence $(1-(1+\theta)\lambda_\epsilon)^{\epsilon^{-1}|y-x|} \to e^{-\mu_\theta|y-x|}$ holds by assumption and

$$\frac{e^{2\mu_\theta^2\delta}}{2}\operatorname{erfc}\left(\frac{4\mu_\theta\delta - |y-x|}{2\sqrt{2\delta}}\right) \to 1,$$

by continuity of the error function, since

$$\frac{4\mu_\theta\delta - |y-x|}{2\sqrt{2\delta}} \leq \frac{4\mu_\theta\delta - \epsilon^\gamma}{2\sqrt{2\delta}} \to -\infty.$$

This completes the proof for $\mu \in (0,1)$.

Finally, consider the case $\mu = 0$. By assumption the discrete initial condition converges $u_0^{(\epsilon)}(x,y) \to 0$. Consequently, the initial condition on $\mathbb{R}^2$ is $u_0(x,y) = 0$, which has solution $u_t(x,y) = 0$, giving the claimed empty point process kernel function $K_t^c(x,y) = 1$ in (4.26). Proving convergence however is subtle since $u_0^{(\epsilon)}(x,y)$ tends to -1 in the tails $|y-x| \to \infty$ for $\lambda_\epsilon \neq 0$ and we cannot apply the developed methods. Since this case is degenerate, we do not dwell on it. □

4.2.3 Exploratory work on further models

We consider scaling limits for three more CARW models. Convergence is not proved, but the limit point processes are identified by their Pfaffian kernels.

Symmetric coalescing and annihilating random walks with Heaviside initial condition. The limit process of real eigenvalues near the (right) edge of the spectrum in the real Ginibre ensemble is Pfaffian (see example 4). To locate the edge kernel $\mathbf{K}_{\text{Edge}}^{\text{Ginibre}}$ with interacting particle systems, we consider the symmetric CARW system $(X_t : t \geq 0)$ from section 4.2.1 with a one-sided initial condition $X_0(x) = 1$ for $x \leq 0$ and $X_0(x) = 0$ for $x > 0$. Since the initial condition is deterministic, theorem 1 gives that X_t is Pfaffian. Comparing with the product Bernoulli example of section 4.2.1, the scaled point process $X^{(\epsilon)}$ is Pfaffian and the only change to the kernel $\mathbf{K}^{(\epsilon)}(x,y)$ is the initial condition

$$K_0^{(\epsilon)}(x,y) = (-\theta)^{\min\{\epsilon^{-1}y,0\}-\min\{\epsilon^{-1}x,0\}}, \qquad \text{for } x \leq y.$$

The limit point process X^c is Pfaffian and the kernel is again given in the form (4.12) with the corresponding continuum initial condition

$$\begin{cases} \partial_t K_t^c(x,y) &= \Delta K_t^c(x,y) & \text{for } x < y,\, t > 0, \\ K_t^c(x,y) &= 1 & \text{for } x = y,\, t > 0, \\ K_0^c(x,y) &= 1(x, y \geq 0) & \text{for } x \leq y. \end{cases} \tag{4.30}$$

To prove that the scaled point processes $X^{(\epsilon)}$ converge to X^c, one must check that condition (4.19) holds for the initial conditions $\mathbf{K}_0^{(\epsilon)}$ and $\mathbf{K}_0^c$ recast on $\epsilon\mathbb{Z}^2$ and $\mathbb{R}^2$, which should follow along similar lines to theorem 5. Recasting the PDE (4.30) to $\mathbb{R}^2$ by (4.16) and using the explicit heat kernel, the function $K_t^c(x,y)$ is given by

$$K_t^c(x,y) = 1 - 2F_2\left(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\right),$$

where F_2 is defined by (2.11). Using proposition 12 to move a factor of $(2t)^{-1/2}$ from $(\mathbf{K}_t^c)_{22}$ to $(\mathbf{K}_t^c)_{11}$ and to swap the order of entries gives an equivalent kernel $\tilde{\mathbf{K}}_t^c(x,y)$, which for $\theta = 1$ satisfies

$$\tilde{\mathbf{K}}_t^c(x,y) = \frac{1}{\sqrt{2t}} \mathbf{K}_{\text{Edge}}^{\text{Ginibre}}\left(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\right).$$

Thus, comparing with (2.12), the mysterious link between annihilating particle systems and real eigenvalues for the real Ginibre ensemble works not only in the bulk but also at the edge of the spectrum.

Absorbed coalescing and annihilating random walks. Consider the CARW process $(X_t : t \geq 0)$ on $\mathbb{Z}$ defined by the following rates and initial conditions

$$q_x = \begin{cases} 1 & \text{for } x \geq 1, \\ 2 & \text{for } x = 0, \\ 0 & \text{for } x < 0, \end{cases} \qquad p_x = \begin{cases} 1 & \text{for } x \geq 1, \\ 0 & \text{for } x < 1, \end{cases} \qquad X_0(x) = \begin{cases} 1 & \text{for } x \geq 0, \\ 0 & \text{for } x < 0, \end{cases}$$

for some $\theta \in [0,1]$. This is a system of coalescing and annihilating simple symmetric random walks on $\{-1, 0, 1, \dots\}$ that are absorbed at the site -1. The anomalous rate q_0 reflects an accelerated rate of absorption $0 \mapsto -1$. This is chosen for technical convenience, as we see below, but we believe that it does not affect the scaling limit of the process. The corresponding one-particle generator L, given by (3.7), is the generator for a reflected random walk on $\{0, 1, 2, \dots\}$, which jumps $x \to x \pm 1$ at rate 1 for $x \geq 1$ and jumps $0 \mapsto 1$ at rate 2. By theorem 1 X_t is Pfaffian with kernel

defined by K_t solving

$$\left\{\begin{array}{rcll} \partial_t K_t(x,y) &=& (L_x+L_y)K_t(x,y) & \text{for } x<y,\, t>0, \\ K_t(x,y) &=& 1 & \text{for } x=y,\, t>0, \\ K_0(x,y) &=& (-\theta)^{\max\{y,0\}-\max\{x,0\}} & \text{for } x\le y. \end{array}\right. \tag{4.31}$$

Note that this does not immediately fit into the developed framework as L is spatially dependent. However, since no particle ever visits $\{\dots,-3,-2\}$ and absorbed particles never escape $\{-1\}$, it is natural to restrict attention to the point process $(X_t(x) : x = 0,1,2,\dots)$. Correspondingly, the discrete PDE (4.31) need only be defined on $\{0 \le x < y\}$. In particular the point process $(X_t(x) : x = 0,1,2,\dots)$ is Pfaffian with kernel defined by $K'_t = K_t|_{0\le x<y}$, which solves

$$\left\{\begin{array}{rcll} \partial_t K'_t(x,y) &=& (\Delta^{(1)}_x+\Delta^{(1)}_y)K'_t(x,y) & \text{for } 0<x<y,\, t>0, \\ \partial_t K'_t(x,y) &=& (2D^+_x+\Delta^{(1)}_y)K'_t(x,y) & \text{for } 0=x<y,\, t>0, \\ K'_t(x,y) &=& 1 & \text{for } 0\le x=y,\, t>0, \\ K'_0(x,y) &=& (-\theta)^{y-x} & \text{for } 0\le x\le y. \end{array}\right. \tag{4.32}$$

The equation does not appear simple, however the $x=0$ differential equation disguises a Neumann boundary condition. To see this, extend to the wedge $\{|x|\le y\}$ by the method of images. In particular, consider the symmetrised extension $\tilde{K}'_t(x,y) = K'_t(|x|,y)$ on $\{|x|\le y\}$. Firstly, note that $\nabla^{(1)}_x \tilde{K}'_t(0,y) = \frac{1}{2}\left(\tilde{K}'_t(1,y) - \tilde{K}'_t(-1,y)\right) = 0$. Moreover

$$\begin{aligned} 2D^+_x K'_t(0,y) &= 2K'_t(1,y) - 2K'_t(0,y) \\ &= \tilde{K}'_t(1,y) + \tilde{K}'_t(-1,y) - 2\tilde{K}'_t(0,y) = \Delta^{(1)}_x \tilde{K}'_t(0,y), \end{aligned}$$

and hence

$$\partial_t \tilde{K}'_t(x,y) = \partial_t K'_t(|x|,y) = (\Delta^{(1)}_x+\Delta^{(1)}_y)\tilde{K}'_t(x,y).$$

All together, the symmetrised function solves

$$\left\{\begin{array}{rcll} \partial_t \tilde{K}'_t(x,y) &=& (\Delta^{(1)}_x+\Delta^{(1)}_y)\tilde{K}'_t(x,y) & \text{for } |x|<y,\, t>0, \\ \nabla^{(1)}\tilde{K}'_t(x,y) &=& 0 & \text{for } |x|<y,\, t>0, \\ \tilde{K}'_t(x,y) &=& 1 & \text{for } |x|=y,\, t>0, \\ \tilde{K}'_0(x,y) &=& (-\theta)^{y-|x|} & \text{for } |x|\le y. \end{array}\right. \tag{4.33}$$

The simple form of this PDE is facilitated by, and justifies, the choice $q_0 = 2$. In particular we can now read off the candidate limit PDE on $\{0 \le x < y\}$.

The point process $(X_t(x) : x = 0, 1, 2, \dots)$ has a scaling limit which is a Pfaffian on $[0, \infty)$ with kernel in the form (4.12) where $(K^c_t(x, y) : t \geq 0, 0 \leq x \leq y)$ solves

$$\begin{cases} \partial_t K^c_t(x,y) &= \Delta K^c_t(x,y) & \text{for } 0 < x < y,\ t > 0, \\ \partial_1 K^c_t(x,y) &= 0 & \text{for } 0 = x < y,\ t > 0, \\ K^c_t(x,y) &= 1 & \text{for } 0 < x = y,\ t > 0, \\ K^c_0(x,y) &= 0 & \text{for } 0 \leq x \leq y. \end{cases}$$

The limit PDE recast on $\mathbb{R}^2$ has initial condition $2\,1(|y| \leq |x|)$, and developing the convolution with the heat kernel gives the explicit solution for K^c_t as

$$K^c_t(x,y) = 1 - \operatorname{erf}\left(\frac{y+x}{2\sqrt{2t}}\right) \operatorname{erf}\left(\frac{y-x}{2\sqrt{2t}}\right).$$

To prove the convergence one must check that condition (4.19) holds.

Reflecting coalescing and annihilating random walks. Consider the CARW process $(X_t : t \geq 0)$ on $\mathbb{Z}$ defined by the following rates and initial conditions

$$q_x = p_x = \begin{cases} 1 & \text{for } x \geq 1, \\ 0 & \text{for } x < 1, \end{cases}, \qquad X_0(x) = \begin{cases} 1 & \text{for } x \geq 0, \\ 0 & \text{for } x < 0, \end{cases}$$

for some $\theta \in [0, 1]$. This is a system of coalescing and annihilating simple symmetric random walks on $\{0, 1, \dots\}$ that are reflected at the origin. As for the absorbed example, we restrict attention to $\{0, 1, \dots\}$. The corresponding one-particle generator L is the generator for a random walk on $\{0, 1, \dots\}$ absorbed at 0. The limiting continuum kernel, for a Pfaffian point process on $[0, \infty)$, is of the form (4.12) where $(K^c_t(x, y) : t \geq 0, 0 \leq x \leq y)$ solves

$$\begin{cases} \partial_t K^c_t(x,y) &= \Delta K^c_t(x,y) & \text{for } 0 < x < y,\ t > 0, \\ K^c_t(x,y) &= 1 & \text{for } 0 < x = y,\ t > 0, \\ K^c_0(x,y) &= 0 & \text{for } 0 \leq x \leq y, \end{cases}$$

with a further Dirichlet boundary condition $K^c_t(0, y) = \Phi_t(y)$ for $y \geq 0$ where Φ solves

$$\begin{cases} \partial_t \Phi_t(y) &= \Delta \Phi_t(y) & \text{for } 0 < y,\ t > 0, \\ \Phi_t(0) &= 1 & \text{for } t > 0, \\ \Phi_0(y) &= 0 & \text{for } 0 \leq y. \end{cases}$$

The boundary condition can be found first for the discrete case, by showing that the expectation $K_t(0,y) = \mathbb{E}_{X_0}\left[(-\theta)^{X_t[0,y)}\right]$ satisfies the analogous discrete equation in t, y. The equations can be solved explicitly and are given in the scaling form

$$K_t^c(x,y) = K_{\frac{1}{4}}^c\left(\frac{x}{2\sqrt{t}}, \frac{y}{2\sqrt{t}}\right),$$

where

$$K_{\frac{1}{4}}^c(x,y) = 1 - \frac{1}{4}\int_x^y\int_y^\infty \mathrm{erf}''\left(\frac{w-z}{\sqrt{2}}\right)\mathrm{erf}\left(\frac{w+z}{\sqrt{2}}\right) + \mathrm{erf}''\left(\frac{w+z}{\sqrt{2}}\right)\mathrm{erf}\left(\frac{w-z}{\sqrt{2}}\right)\,\mathrm{d}z\mathrm{d}w.$$

Asymmetric rates and independent initial conditions. Consider the CARW process with homogeneous, but not necessarily symmetric, rates $p_x = p > 0$, $q_x = q > 0$ for $x \in \mathbb{Z}$, interaction parameter $\theta \in [0,1]$ and initial condition independent Bernoulli(λ) for $\lambda \in (0,1]$. The one-particle generator is given by (3.7)

$$L^{(1)}f(x) = \frac{q+p}{2}\Delta^{(1)}f(x) + (q-p)\nabla^{(1)}f(x),$$

for $f : \mathbb{Z} \to \mathbb{R}$, where the discrete operators are defined by

$$\begin{aligned} \nabla^{(\epsilon)}f(x) &= \left(f(x+\epsilon) - f(x-\epsilon)\right)/(2\epsilon), \\ \Delta^{(\epsilon)}f(x) &= \epsilon^{-2}\left(f(x+\epsilon) + f(x-\epsilon) - 2f(x)\right). \end{aligned} \tag{4.34}$$

As for individual random walks, for the scaled point processes to converge to a non-degenerate limit the asymmetry must be scaled. To construct the scaled processes, we first define a sequence of models on $\mathbb{Z}$ with ϵ-dependent rates. For $\epsilon > 0$ the process $(X_t^{[\epsilon]} : t \geq 0)$ is defined to be a CARW process on $\mathbb{Z}$ with weakly asymmetric rates $p_x^{(\epsilon)} = p^{(\epsilon)} > 0$ and $q_x^{(\epsilon)} = q^{(\epsilon)} > 0$ for $x \in \mathbb{Z}$, with

$$p^{(\epsilon)} = \frac{q+p}{2} - \frac{q-p}{2}\epsilon, \qquad q^{(\epsilon)} = \frac{q+p}{2} + \frac{q-p}{2}\epsilon,$$

and initial condition independent Bernoulli(λ_ϵ) for $\lambda_\epsilon \in [0,1]$. For fixed $t > 0$, the scaled processes $X^{(\epsilon)}$ on $\epsilon\mathbb{Z}$ are then given by scaling diffusively

$$X^{(\epsilon)}(\mathrm{d}x) = X_{\epsilon^{-2}t}^{[\epsilon]}(\epsilon^{-1}\mathrm{d}x) \qquad \text{on } \epsilon\mathbb{Z}. \tag{4.35}$$

The point process $X^{(\epsilon)}$ is Pfaffian with kernel of the form (4.10) in terms of the kernel function $K_t^{(\epsilon)}$ solving

$$\begin{cases} \partial_t K_t^{(\epsilon)}(x,y) &= (L_x^{(\epsilon)} + L_y^{(\epsilon)}) K_t^{(\epsilon)}(x,y) & \text{for } x < y,\ t > 0, \\ K_t^{(\epsilon)}(x,x) &= 1 & \text{for all } x,\ t > 0, \\ K_0^{(\epsilon)}(x,y) &= (1-(1+\theta)\lambda)^{\epsilon^{-1}(y-x)} & \text{for } x \le y, \end{cases} \tag{4.36}$$

where the one-particle generator $L^{(\epsilon)}$ is given, for $f : \epsilon\mathbb{Z} \to \mathbb{R}$, by

$$\begin{aligned} L^{(\epsilon)} f(x) &= \frac{q^{(\epsilon)} + p^{(\epsilon)}}{2} \Delta^{(\epsilon)} f(x) + \epsilon^{-1}(q^{(\epsilon)} - p^{(\epsilon)}) \nabla^{(\epsilon)} f(x) \\ &= \frac{q+p}{2} \Delta^{(\epsilon)} f(x) + (q-p) \nabla^{(\epsilon)} f(x). \end{aligned}$$

We can read off the candidate limit PDE. The only difference to (4.25) is the one-particle generator, given, for $f : \mathbb{R} \to \mathbb{R}$, by

$$L^c f(x) = \frac{q+p}{2} \partial_x^2 f(x) + (q-p) \partial_x f(x).$$

The solution is given by (4.26), the solution to (4.25), with $t \mapsto \frac{q+p}{2} t$. Indeed the additional first order terms cancel, $(\partial_x + \partial_y) K_t^c(x,y) = 0$, since $K_t^c(x,y)$ depends on $y - x$. The single solution makes sense because choosing asymmetric homogeneous rates corresponds to time-scaling the original symmetric CARW process (as in remark 14) and adding an (ϵ-dependent) drift to particles. Since the point processes are spatially homogeneous, a global shift does not change the distribution of particles. In fact, the same limit point process should emerge for general weakly asymmetric rates, where

$$p^{(\epsilon)} = \frac{q+p}{2} + c_1 \frac{q-p}{2} \epsilon + O(\epsilon^2), \quad q^{(\epsilon)} = \frac{q+p}{2} + c_2 \frac{q-p}{2} \epsilon + O(\epsilon^2),$$

for c_1, $c_2 \in \mathbb{R}$ with $c_2 - c_1 = 1$. Note that $p^{(\epsilon)} > 0$ and $q^{(\epsilon)} > 0$ for sufficiently small ϵ.

To prove convergence of scaled asymmetric CARW, we cannot simply follow the proof of theorem 6 because the drift plays a role when recasting the PDEs on the whole space. The resultant equation does not have constant coefficients and so falls outside the scope of the derived convergence condition. We believe, however, that the analogous estimates for half-space PDEs remain standard.

Remark 18. Convergence of the characterising PDEs in the general case of spatially-dependent coefficients should hold under some conditions on the coefficients and

their derivatives. The limit PDE is clear but the proof will necessarily be more involved.

4.3 Heuristics for limits of annihilating random walks with pairwise immigration

Scaling limits for the ARWI model of section 3.3.1 are developed in some simple cases, building on the work of section 4.2.

As for the construction of the discrete models, the scaling theory for ARWI is a direct extension of CARW in the case $\theta = 1$. The only difference is the addition of a potential term in the characterising PDEs. This is enough, however, to take us outside the scope of the whole-space PDE estimates of appendix B. We consider in section 4.3.1 convergence of the scaled point processes for homogeneous rates and scaled independent initial conditions. Unlike CARW, for ARWI there is a non-degenerate steady state continuum point process under the additional limit $t \to \infty$, computed in section 4.3.2. Finally, in section 4.3.3 we consider a natural spatially inhomogeneous model and its limit, the Brownian Firework.

4.3.1 Homogeneous rates and scaled independent initial conditions

Consider the ARWI process $(X_t : t \geq 0)$ on $\mathbb{Z}$ with homogeneous rates $p_x = p > 0$, $q_x = q > 0$ and $m_x = m \geq 0$ for $x \in \mathbb{Z}$, and independent Bernoulli(λ) initial condition for some $\lambda \in (0, 1]$. Theorem 2 and remark 6 give that X_t is Pfaffian and provide an expression for the kernel. Fixing rates $p^{(\epsilon)} > 0$, $q^{(\epsilon)} > 0$, $m^{(\epsilon)} \geq 0$ and $\lambda_\epsilon \in [0, 1]$ for $\epsilon > 0$, the scaled point process $X^{(\epsilon)}$ on $\epsilon\mathbb{Z}$ is defined by diffusively scaling the ARWI process with these ϵ-dependent rates, i.e. scaling via (4.35). It follows that $X^{(\epsilon)}$ is Pfaffian with kernel $\mathbf{K}^{(\epsilon)}(x, y)$ given, for $x < y$, by

$$\mathbf{K}^{(\epsilon)}(x,y) = -\frac{\epsilon}{2}\begin{pmatrix} K_t^{(\epsilon)}(x,y) & D_\epsilon^{(0,1)}K_t^{(\epsilon)}(x,y) \\ D_\epsilon^{(1,0)}K_t^{(\epsilon)}(x,y) & D_\epsilon^{(1,1)}K_t^{(\epsilon)}(x,y) \end{pmatrix}, \tag{4.37}$$

and $\mathbf{K}_{12}^{(\epsilon)}(x, x) = -\frac{\epsilon}{2} D_\epsilon^{(0,1)} K_t^{(\epsilon)}(x, x)$. The function $(K_t^{(\epsilon)}(x, y) : t \geq 0, x, y \in \epsilon\mathbb{Z}, x < y)$ is the unique bounded solution to

$$\begin{cases} \partial_t K_t^{(\epsilon)}(x,y) &= (L_x^{(\epsilon)} + L_y^{(\epsilon)})K_t^{(\epsilon)}(x,y) & \text{for } x < y,\ t > 0, \\ K_t^{(\epsilon)}(x,x) &= 1 & \text{for all } x,\ t > 0, \\ K_0^{(\epsilon)}(x,y) &= (1 - 2\lambda_\epsilon)^{\epsilon^{-1}(y-x)} & \text{for } x \leq y, \end{cases} \tag{4.38}$$

where the one-particle operator $L^{(\epsilon)}$ is given, for $f : \epsilon\mathbb{Z} \to \mathbb{R}$ and in terms of the operators (4.34), by

$$L^{(\epsilon)}f(x) = \frac{q^{(\epsilon)} + p^{(\epsilon)}}{2}\Delta_x^{(\epsilon)}f(x) + \epsilon^{-1}(q^{(\epsilon)} - p^{(\epsilon)})\nabla_x^{(\epsilon)}f(x) - \epsilon^{-2}2m^{(\epsilon)}f(x). \quad (4.39)$$

Note that for $\epsilon = 1$ this is equivalent to (3.21). With lemma 8 in mind, the candidate continuum kernel is given by the limit of $\epsilon^{-1}\mathbf{K}^{(\epsilon)}(x, y)$ and we hence consider convergence of $K_t^{(\epsilon)}(x, y)$ and its discrete derivatives. In order for the PDE solutions to converge, the one-particle generators $L^{(\epsilon)}$ should converge and this informs the appropriate scaling of rates. As for the asymmetric CARW example above, we take weakly asymmetric jump rates $q^{(\epsilon)}$ and $p^{(\epsilon)}$. To avoid immigration swamping the limit, the rate $m^{(\epsilon)}$ must be dampened and balance with annihilation. All together, fixing $a > 0$, $b \in \mathbb{R}$, $c \geq 0$ and c_1, $c_2 \in \mathbb{R}$ with $c_2 - c_1 = 1$, we pick rates

$$p^{(\epsilon)} = \frac{a}{2} + c_1\frac{b}{2}\epsilon + O(\epsilon^2), \quad q^{(\epsilon)} = \frac{a}{2} + c_2\frac{b}{2}\epsilon + O(\epsilon^2), \quad m^{(\epsilon)} = \frac{c}{2}\epsilon^2 + O(\epsilon^3). \quad (4.40)$$

The candidate limit one-particle generator is then given, for $f : \mathbb{R} \to \mathbb{R}$, by

$$L^c f(x) = a\partial_x^2 f(x) + b\partial_x f(x) - cf(x). \quad (4.41)$$

As for CARW, the initial condition of the limit is determined by the convergence of $\epsilon^{-1}\lambda_\epsilon$. We arrive at the following conjecture.

Conjecture 1. *Fix $t > 0$, $a > 0$, $b \in \mathbb{R}$, $c \geq 0$ and $\lambda_\epsilon \in [0, 1]$ for $\epsilon > 0$. Then the scaled ARWI point process $X^{(\epsilon)}$ with rates* (4.40) *and initial condition independent Bernoulli(λ_ϵ) converges in distribution to X^c, the Pfaffian point process on $\mathbb{R}$ with kernel $\mathbf{K}^c(x, y)$, given, for $x < y$, by*

$$\mathbf{K}^c(x, y) = -\frac{1}{2}\begin{pmatrix} K_t^c(x, y) & \partial_2 K_t^c(x, y) \\ \partial_1 K_t^c(x, y) & \partial_1\partial_2 K_t^c(x, y) \end{pmatrix}, \quad (4.42)$$

and $\mathbf{K}_{12}^c(x, x) = -\frac{1}{2}\partial_2 K_t^c(x, x)$, where $(K_t^c(x, y) : t \geq 0,\ x, y \in \mathbb{R},\ x < y)$ is the unique bounded solution to

$$\begin{cases} \partial_t K_t^c(x, y) &= (L_x^c + L_y^c)K_t^c(x, y) & \text{for } x < y,\ t > 0, \\ K_t^c(x, x) &= 1 & \text{for all } x,\ t > 0, \\ K_0^c(x, y) &= e^{-2\mu(y-x)} & \text{for } x \leq y, \end{cases} \quad (4.43)$$

with L^c given by (4.41) *and $\mu = \lim_{\epsilon\downarrow 0}\epsilon^{-1}\lambda_\epsilon \in [0, \infty]$. In particular, $K_t^c(x, y)$ may*

be written explicitly, for $x < y$ and $t \geq 0$, as

$$K_t^c(x,y) = \frac{2}{\sqrt{\pi}} \int_{\alpha/\sqrt{t}}^{\infty} e^{-w^2} e^{-2c\alpha^2/w^2} \, \mathrm{d}w + e^{-2ct}\big(F_{2\mu,at}(y-x) - F_{2\mu,at}(x-y)\big), \quad (4.44)$$

where $\alpha = \frac{y-x}{2\sqrt{2a}}$ and $F_{2\mu,at}(z)$ is given by (4.27).

Remark 19. Even in the symmetric case, we cannot simply recycle the proofs of theorems 5 and 6. Indeed, due to the additional potential term, the PDE (4.43) recast on $\mathbb{R}^2$ only has piecewise continuous second derivatives and the derived estimates do not apply. As for asymmetric CARW, we believe that the corresponding estimates for half-space PDEs remain standard.

To see that (4.44) is the solution to (4.43), we first force Dirichlet boundary conditions by considering $u_t(x,y) = K_t^c(x,y) - 1$, which solves

$$\begin{cases} \partial_t u_t(x,y) &= (L_x^c + L_y^c) u_t(x,y) - 2c & \text{for } x < y,\ t > 0, \\ u_t(x,x) &= 0 & \text{for all } x,\ t > 0, \\ u_0(x,y) &= e^{-2\mu(y-x)} - 1 & \text{for } x \leq y. \end{cases} \quad (4.45)$$

Using linearity we split into a homogeneous and an inhomogeneous PDE, writing $u_t(x,y) = u_t^{(1)}(x,y) + u_t^{(2)}(x,y)$, where

$$\begin{cases} \partial_t u_t^{(1)}(x,y) &= (L_x^c + L_y^c) u_t^{(1)}(x,y) & \text{for } x < y,\ t > 0, \\ u_t^{(1)}(x,x) &= 0 & \text{for all } x,\ t > 0, \\ u_0^{(1)}(x,y) &= e^{-2\mu(y-x)} - 1 & \text{for } x \leq y, \end{cases} \quad (4.46)$$

and

$$\begin{cases} \partial_t u_t^{(2)}(x,y) &= (L_x^c + L_y^c) u_t^{(2)}(x,y) - 2c & \text{for } x < y,\ t > 0, \\ u_t^{(2)}(x,x) &= 0 & \text{for all } x,\ t > 0, \\ u_0^{(2)}(x,y) &= 0 & \text{for } x \leq y. \end{cases} \quad (4.47)$$

The solution to (4.46) is given by inspecting the proofs of theorems 5 and 6. Indeed the $a = 1$, $b = 0$, $c = 0$, $\mu = \infty$ case is given in theorem 5 by (4.21), upon restricting to the set $\{x \leq y\}$. Theorem 6 shows that for $\mu \in (0, \infty)$ there is an additional term (4.29). As in the final example of section 4.2.3, general $a > 0$ corresponds to a time-scaling $t \mapsto at$ and the solutions are all independent of any drift $b \in \mathbb{R}$. The only difference to the PDE for $c > 0$ is the additional potential term $-2cu_t^{(1)}(x,y)$. This translates into an exponential factor and pulling everything together we arrive

at

$$u_t^{(1)}(x,y) = e^{-2ct}\operatorname{erf}\left(\frac{x-y}{2\sqrt{2at}}\right) + e^{-2ct}\big(F_{2\mu,at}(y-x) - F_{2\mu,at}(x-y)\big).$$

Consider now the inhomogeneous equation (4.47) with driving term. Firstly, the homogeneous solution above gives the solution to the auxiliary equation

$$\begin{cases} \partial_t \tilde{u}_t^{(2)}(x,y) &= (L_x^c + L_y^c)\tilde{u}_t^{(2)}(x,y) & \text{for } x<y,\, t>0, \\ \tilde{u}_t^{(2)}(x,x) &= 0 & \text{for all } x,\, t>0, \\ \tilde{u}_0^{(2)}(x,y) &= -2c & \text{for } x \le y. \end{cases} \tag{4.48}$$

Using Duhamel's principle $u_t^{(2)}(x,y)$ may then be expressed as follows

$$u_t^{(2)}(x,y) = \int_0^t \tilde{u}_s^{(2)}(x,y)\,\mathrm{d}s = 2c\int_0^t e^{-2cs}\operatorname{erf}\left(\frac{x-y}{2\sqrt{2as}}\right)\mathrm{d}s. \tag{4.49}$$

All together the limit kernel is given by

$$K_t^c(x,y) = 1 + e^{-2ct}\operatorname{erf}\left(\frac{x-y}{2\sqrt{2at}}\right) + 2c\int_0^t e^{-2cs}\operatorname{erf}\left(\frac{x-y}{2\sqrt{2as}}\right)\mathrm{d}s \\ + e^{-2ct}\big(F_{2\mu,at}(y-x) - F_{2\mu,at}(x-y)\big).$$

Integration by parts simplifies the first three terms

$$1 + e^{-2ct}\operatorname{erf}\left(\frac{x-y}{2\sqrt{2at}}\right) + 2c\int_0^t e^{-2cs}\operatorname{erf}\left(\frac{x-y}{2\sqrt{2as}}\right)\mathrm{d}s \\ = \int_0^t e^{-2cs}\left(\partial_s \operatorname{erf}\left(\frac{x-y}{2\sqrt{2as}}\right)\right)\mathrm{d}s,$$

and the claimed expression (4.44) is then obtained upon evaluating the derivative and changing variables with $s \mapsto \left(\frac{x-y}{2\sqrt{2a}}\right)^2 w^{-2}$.

4.3.2 Large-time limit for constant rates

The number of particles in the CARW process decreases as time evolves and the process is degenerate in the limit as $t \to \infty$. For ARWI, however, the number of particles increases due to immigration, and balancing with annihilation there is a chance for a non-degenerate steady state. The above scaling regime for ARWI balances immigration with annihilation and we now investigate the behaviour of the continuum point processes at large times. With lemma 9 in mind, consider the limit as $t \to \infty$ of the kernel function $K_t^c(x,y)$ in (4.44), denoted by $K_\infty^c(x,y)$. This

should solve the elliptic version of (4.43), namely

$$\begin{cases} 0 &= (L^c_x + L^c_y)K^c_\infty(x,y) & \text{for } x < y, \\ K^c_\infty(x,x) &= 1 & \text{for all } x, \end{cases}$$

where L^c is given by (4.41). The unique bounded solution is $K^c_\infty(x,y) = e^{-\sqrt{\frac{c}{a}}(y-x)}$. Indeed we recover this expression by taking limits in (4.44). The large-time limit point process is Pfaffian with corresponding kernel of the form (4.42). Arguing as for the initial condition of theorem 6, by example 2 and proposition 15 this point process is a rate $\frac{1}{2}\sqrt{\frac{c}{a}}$ Poisson process. Moreover this distribution is invariant in the sense that we recover the solution $e^{-\sqrt{\frac{c}{a}}(y-x)}$ upon substituting $\mu = \frac{1}{2}\sqrt{\frac{c}{a}}$ into (4.44). Indeed the time derivative of $K^c_t(x,t)$ vanishes for this choice of μ, then picking $t = 0$ or $t = \infty$, for example, gives the desired expression. We expect that there are underlying stochastic processes with one-dimensional marginals given by our Pfaffian scaling limits and the large-time limit should correspond to the unique steady state. It is reasonable that for $c > 0$ there is a single steady state, since the influence of the initial condition should quickly dissipate due to infinitesimal pairwise immigration and annihilation. Note that for $c = 0$ the steady state is the degenerate empty point process, highlighting that the corresponding analysis is not interesting for CARW.

4.3.3 Exploratory work on the Brownian Firework

We discuss a particular ARWI model, investigating the scaling limit via some formal calculations. The motivation is to better understand the infinitesimal pairwise immigration mechanism in the continuum. The underlying model $(X_t : t \geq 0)$ on $\mathbb{Z}$ is defined by taking constant jump rates $p_x = q_x = 1/2$ for $x \in \mathbb{Z}$. It remains to define the immigration parameter and initial conditions. To single out the immigration mechanism we consider the case of empty initial conditions and $m_x = \frac{m}{2}1(x = 0)$ for some $m > 0$. Thus, particles can only enter the system via immigration 'at the origin', i.e. as a pair onto the sites $\{-1, 0\}$. It is interesting to understand if there is a scaling limit of this process and how the balance between annihilation and immigration manifests itself. For example, what is the typical dispersion of particles? How does the parameter m enter the distribution? What are the large-time asymptotics? In the continuum limit, the immigration occurs at the origin where there is a constant explosion of infinitesimal pairs entering the system but mostly annihilating with each other. We have come to call the model the *Brownian Firework*.

We consider scaled rates $p_x^{(\epsilon)} = q_x^{(\epsilon)} = 1/2$ and $m_x^{(\epsilon)} = \frac{m}{2}\epsilon 1(x = 0)$ for $x \in \epsilon\mathbb{Z}$, then the one-particle generator is given, for $f : \epsilon\mathbb{Z} \to \mathbb{R}$, by

$$L^{(\epsilon)} f(x) = \frac{1}{2}\Delta_x^{(\epsilon)} f(x) - m\epsilon^{-1} 1(x = 0) f(x).$$

Note that the immigration rate is scaled differently to the homogeneous case. This is in order to see a non-degenerate contribution from immigration in the continuum limit. By theorem 2 the scaled point process $X^{(\epsilon)}$ in (4.6) is Pfaffian with kernel in the form (4.37), where $(K_t^{(\epsilon)}(x, y) : t \geq 0, x \leq y)$ solves (4.38) with initial condition $K_0^{(\epsilon)}(x, y) = 1$ for $x \leq y$. Formally the continuum limit is given by

$$\begin{cases} \partial_t K_t^c(x,y) &= \Big(\frac{1}{2}\Delta - m\big(\delta_0(x) + \delta_0(y)\big)\Big) K_t^c(x,y) & \text{for } x < y,\ t > 0, \\ K_t^c(x,x) &= 1 & \text{for all } x,\ t > 0, \\ K_0^c(x,y) &= 1 & \text{for } x \leq y. \end{cases} \tag{4.50}$$

To find an explicit expression for $K_t^c(x, y)$, consider $g_t(z, w) : \mathbb{R}^2 \to [0, \infty)$ defined by

$$g_t(z, w) = \mathbb{E}_z \left[\delta_0(B_t) e^{-mL_t^w}\right] = \mathbb{E}_{z-w} \left[\delta_{-w}(B_t) e^{-mL_t^0}\right],$$

where $(B_t : t \geq 0)$ is a Brownian motion with $B_0 = z$ and L_t^w denotes the local time at level w up to time t. For fixed $w \in \mathbb{R}$, the Feynman-Kac formula implies that $g_t(z, w)$ solves

$$\begin{cases} \partial_t g_t(z,w) &= \Big(\frac{1}{2}\Delta_z - m\delta_w(z)\Big) g_t(z,w) & \text{for } z \in \mathbb{R},\ t > 0, \\ g_0(z,w) &= \delta_0(z) & \text{for } z \in \mathbb{R}. \end{cases}$$

One way to see this is to solve the discrete equation with the discrete Feynman-Kac formula and note that the term in the exponential converges to Brownian local time. This illustrates the appropriate scaling in the immigration parameter $m_x^{(\epsilon)}$. The solution to (4.50) is then given by

$$K_t^c(x, y) = 1 - m \int_0^t h_s(x, y)\, \mathrm{d}s,$$

where

$$h_t(x, y) = \int_{\mathbb{R}^2} g_s(x - w_1, -w_1) g_s(y - w_2, -w_2)(\delta_0(w_1) + \delta_0(w_2)) \operatorname{sgn}(w_2 - w_1)\, \mathrm{d}w_1 \mathrm{d}w_2.$$

Using the explicit joint distribution for Brownian motion and its local time [33] we find

$$g_t(z-w,-w) = \frac{1}{\sqrt{2\pi t}}\left(e^{-\frac{(z-w)^2}{2t}} - e^{-\frac{(z+w)^2}{2t}}\right)\mathbb{I}_{\{zw>0\}} + e^{m|w|+m|z|}\int_{|w|+|z|}^{\infty} e^{-mu}\frac{u}{\sqrt{2\pi t^3}}e^{-\frac{u^2}{2t}}\,\mathrm{d}u.$$

Consider now the large-time asymptotics. With lemma 9 in mind, we take limits in the kernel function $K^c_\infty(x,y) = \lim_{t\to\infty} K^c_t(x,y)$. A calculation reveals

$$K^c_\infty(x,y) = 1 + \frac{2m}{\pi}\int_0^\infty e^{-mu}\left(\arctan\left(\frac{x}{z+|y|}\right) - \arctan\left(\frac{y}{z+|x|}\right)\right)\mathrm{d}z,$$

for $x \le y$. Let $\Phi : \mathbb{R}^2 \to \mathbb{R}$ be a smooth test function with compact support satisfying the boundary condition $\Phi(x,x) = 0$. It can be checked that K^c_∞ weakly satisfies the elliptic form of (4.50)

$$\iint_{x<y} K^c_\infty(x,y)\frac{1}{2}\Delta\Phi(x,y)\,\mathrm{d}x\mathrm{d}y = \iint_{x<y} m\left(\delta_0(x)+\delta_0(y)\right)K^c_\infty(x,y)\Phi(x,y)\,\mathrm{d}x\mathrm{d}y.$$

The kernel function simplifies further in the limit $m \to \infty$

$$\lim_{m\to\infty} K^c_\infty(x,y) = \begin{cases} 1 + \frac{2}{\pi}\left(\arctan(\frac{x}{y}) - \arctan(\frac{y}{x})\right) & \text{if } 0 < x < y, \\ 0 & \text{if } x < 0 < y, \\ 1 + \frac{2}{\pi}\left(\arctan(\frac{y}{x}) - \arctan(\frac{x}{y})\right) & \text{if } x < y < 0. \end{cases}$$

Computing derivatives, the kernel $\mathbf{K}_\infty$ for the $t \to \infty$, $m \to \infty$ point process is given, for $x < y$, by

$$\mathbf{K}_\infty(x,y) = \begin{pmatrix} -\frac{1}{2} - \frac{1}{\pi}\left(\arctan\left(\frac{x}{|y|}\right) - \arctan\left(\frac{y}{|x|}\right)\right) & \frac{x}{\pi}\frac{\mathrm{sgn}(x)+\mathrm{sgn}(y)}{x^2+y^2} \\ -\frac{y}{\pi}\frac{\mathrm{sgn}(x)+\mathrm{sgn}(y)}{x^2+y^2} & \frac{1}{\pi}\frac{\mathrm{sgn}(x)+\mathrm{sgn}(y)}{(x^2+y^2)^2}(y^2-x^2) \end{pmatrix},$$

and $(\mathbf{K}_\infty)_{12}(x,x) = \frac{1}{\pi|x|}$. The one-point intensity is $\rho^{(1)}(x) = \frac{1}{\pi|x|}$, suggesting that there may be an accumulation point at the origin. Note that $\mathbf{K}_\infty(x,y) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ for $x < 0 < y$, then using proposition 10 the intensity for $x_1, \ldots, x_n < 0$ and $y_1, \ldots, y_m > 0$ factorises

$$\rho^{(n+m)}(x_1,\ldots,x_n,y_1,\ldots,y_m) = \rho^{(n)}(x_1,\ldots,x_n)\rho^{(m)}(y_1,\ldots,y_m).$$

The infinite strength firework of particles at the origin leads to the two half spaces being independent.

4.4 Heuristics for limits of branching coalescing random walks

Scaling limits of the BCRW model from section 3.3.2 are developed, building on the preceding theory for CARW and ARWI.

This extension to the CARW scaling theory of section 4.2 is more subtle than the case of ARWI, due to the pervasion of ϕ factors and the delicate conditions on the rates. However, with section 3.3.3 in mind, it is not surprising that the scaled BCRW processes are related to scaled ARWI for certain initial conditions. This connection allows us to derive convergence as a direct consequence of conjecture 1 and leads to a continuum relation.

Convergence of the scaled BCRW point processes for homogeneous rates and scaled independent initial conditions is considered in section (4.4.1). The relation between the BCRW and ARWI continuum scaling limits is given in section 4.4.2.

4.4.1 Homogeneous rates and scaled independent conditions

Consider the BCRW process $(X_t : t \geq 0)$ with homogeneous rates $p_x = p > 0$, $q_x = q > 0$, $\ell_x = \ell$ and $r_x = r$ for $x \in \mathbb{Z}$ with $p\ell = qr$, and Bernoulli(λ) initial conditions for some $\lambda \in (0, 1]$. Note that conditions (3.29) are satisfied. It follows from theorem 3, and the extension to random initial conditions, that X_t is a Pfaffian point process on $\mathbb{Z}$. Fixing rates $p^{(\epsilon)} > 0$, $q^{(\epsilon)} > 0$, $\ell^{(\epsilon)} \geq 0$, $r^{(\epsilon)} \geq 0$ satisfying $p^{(\epsilon)}\ell^{(\epsilon)} = q^{(\epsilon)}r^{(\epsilon)}$ and $\lambda_\epsilon \in [0, 1]$ for $\epsilon > 0$, the scaled point process $X^{(\epsilon)}$ on $\epsilon\mathbb{Z}$ is defined, as in (4.35), by diffusive scaling and is Pfaffian with kernel $\tilde{\mathbf{K}}^{(\epsilon)}(x, y)$ given, for $x < y$, by

$$\tilde{\mathbf{K}}^{(\epsilon)}(x, y) = -\frac{\epsilon}{\phi^{(\epsilon)}} \begin{pmatrix} \tilde{K}_t^{(\epsilon)}(x, y) & D_\epsilon^{(0,1)} \tilde{K}_t^{(\epsilon)}(x, y) \\ D_\epsilon^{(1,0)} \tilde{K}_t^{(\epsilon)}(x, y) & D_\epsilon^{(1,1)} \tilde{K}_t^{(\epsilon)}(x, y) \end{pmatrix}, \tag{4.51}$$

and $\tilde{\mathbf{K}}_{12}^{(\epsilon)}(x, x) = 1 - \frac{1}{\phi^{(\epsilon)}} \tilde{K}_t^{(\epsilon)}(x, x + \epsilon)$, where

$$\phi^{(\epsilon)} = \sqrt{1 + \frac{\ell^{(\epsilon)}}{q^{(\epsilon)}}} = \sqrt{1 + \frac{r^{(\epsilon)}}{p^{(\epsilon)}}}.$$

The function $(\tilde{K}_t^{(\epsilon)}(x,y) : t \geq 0,\, x, y \in \epsilon\mathbb{Z},\, x < y)$ is the unique solution of exponential growth to

$$\begin{cases} \partial_t \tilde{K}_t^{(\epsilon)}(x,y) &= (\tilde{L}_x^{(\epsilon)} + \tilde{L}_y^{(\epsilon)})\tilde{K}_t^{(\epsilon)}(x,y) & \text{for } x < y,\, t > 0, \\ \tilde{K}_t^{(\epsilon)}(x,x) &= 1 & \text{for all } x,\, t > 0, \\ \tilde{K}_0^{(\epsilon)}(x,y) &= \left(\phi^{(\epsilon)}(1-\lambda_\epsilon)\right)^{\epsilon^{-1}(y-x)} & \text{for } x \leq y, \end{cases} \tag{4.52}$$

where the one-particle operator $\tilde{L}^{(\epsilon)}$ is given, for $f : \epsilon\mathbb{Z} \to \mathbb{R}$, by

$$\tilde{L}^{(\epsilon)} f(x) = \frac{q^{(\epsilon)} + p^{(\epsilon)}}{2}\phi^{(\epsilon)}\Delta_x^{(\epsilon)} f(x) + \frac{q^{(\epsilon)} - p^{(\epsilon)}}{\epsilon}\phi^{(\epsilon)}\nabla_x^{(\epsilon)} f(x) - \frac{q^{(\epsilon)} + p^{(\epsilon)}}{2\epsilon^2}\left(1 - \phi^{(\epsilon)}\right)^2 f(x). \tag{4.53}$$

Note that this is equivalent to (3.36) when $\epsilon = 1$. As for the underlying $\mathbb{Z}$ models, the one-particle generators for the scaled ARWI and BCRW point processes are of the same form and in particular the relations of section 3.3.3 have analogues on $\epsilon\mathbb{Z}$. In section 4.4.2 we give a continuum relation between the scaling limits. We now turn to the rates. Since the CRW model is recovered by setting the branching rates to zero, we should pick weakly asymmetric jump rates. The branching rates must be picked to (asymptotically) balance the ϵ^{-2} factor in the potential term coefficient. Condition (3.29a) implies that they should be equal up to their leading order of ϵ (see remark 20). For definiteness we set the following explicit rates

$$p^{(\epsilon)} = a, \qquad q^{(\epsilon)} = a + \epsilon b, \qquad r^{(\epsilon)} = 2\epsilon\sqrt{ac}, \qquad \ell^{(\epsilon)} = \frac{q^{(\epsilon)} r^{(\epsilon)}}{p^{(\epsilon)}} = 2\epsilon\sqrt{ac}\left(1 + \frac{\epsilon b}{a}\right), \tag{4.54}$$

for some $a > 0$, $b \in \mathbb{R}$ and $c \geq 0$. Note that conditions (3.29) are satisfied by construction and $p^{(\epsilon)} > 0$, $q^{(\epsilon)} > 0$, $r^{(\epsilon)} \geq 0$ and $\ell^{(\epsilon)} \geq 0$ for sufficiently small ϵ. A Taylor expansion of $\phi^{(\epsilon)}$ gives

$$\phi^{(\epsilon)} = \left(1 + 2\epsilon\sqrt{\frac{c}{a}}\right)^{\frac{1}{2}} = 1 + \epsilon\sqrt{\frac{c}{a}} + O(\epsilon^2), \tag{4.55}$$

so that the potential coefficient satisfies

$$\frac{q^{(\epsilon)} + p^{(\epsilon)}}{2\epsilon^2}\left(1 - \phi^{(\epsilon)}\right)^2 = \frac{2a + \epsilon b}{2a}c + O(\epsilon).$$

All together, as $\epsilon \downarrow 0$ the coefficients of $\tilde{L}^{(\epsilon)}$ converge as follows

$$\frac{q^{(\epsilon)}+p^{(\epsilon)}}{2}\phi^{(\epsilon)} \to a, \quad \frac{q^{(\epsilon)}-p^{(\epsilon)}}{\epsilon}\phi^{(\epsilon)} \to b, \quad \frac{q^{(\epsilon)}+p^{(\epsilon)}}{2\epsilon^2}\left(1-\phi^{(\epsilon)}\right)^2 \to c, \tag{4.56}$$

and the candidate limit one-particle generator is given by (4.41). As for CARW and ARWI, the limit initial condition depends on the convergence of $\epsilon^{-1}\lambda_\epsilon$, however there is also a contribution from branching. Indeed using the above expansion

$$\phi^{(\epsilon)}(1-\lambda_\epsilon) = \left(1+\epsilon\sqrt{\frac{c}{a}}+O(\epsilon^2)\right)(1-\lambda_\epsilon) = 1+\epsilon\left(\sqrt{\frac{c}{a}}-\epsilon^{-1}\lambda_\epsilon\right)+\lambda_\epsilon O(\epsilon),$$

and we see that the initial condition of (4.52) satisfies

$$\tilde{K}_0^{(\epsilon)}(x,y) \to e^{-\left(\mu-\sqrt{\frac{c}{a}}\right)(y-x)},$$

where $\mu = \lim_{\epsilon\downarrow 0}\epsilon^{-1}\lambda_\epsilon$. All together, the limit PDE is given by

$$\begin{cases} \partial_t \tilde{K}_t^c(x,y) &= (L_x^c + L_y^c)\tilde{K}_t^c(x,y) & \text{for } x<y,\, t>0, \\ \tilde{K}_t^c(x,x) &= 1 & \text{for all } x,\, t>0, \\ \tilde{K}_0^c(x,y) &= e^{-\left(\mu-\sqrt{\frac{c}{a}}\right)(y-x)} & \text{for } x \le y. \end{cases} \tag{4.57}$$

For $\mu > \sqrt{c/a}$ the initial condition corresponds to a Poisson point process of rate $\mu - \sqrt{c/a} > 0$ and, upon making the transformation $\mu - \sqrt{c/a} \mapsto 2\mu$, the PDE coincides with (4.43) for the ARWI limit. Explicitly, $\tilde{K}_t^c(x,y)$ is given by (4.44) with μ replaced by $\frac{1}{2}\left(\mu - \sqrt{\frac{c}{a}}\right) > 0$. We revisit this connection in section 4.4.2, where we use the PDE equivalence to derive a relation between the ARWI and BCRW limit processes. Note that, perhaps surprisingly, increasing the branching parameter c decreases the Poisson rate. In effect, at small times, more branching leads to more coalescences and on average a net decrease in the number of particles. The case $\mu = \sqrt{c/a}$ corresponds to the initial condition $\tilde{K}_0^c(x,y) = 1$ and we may express $\tilde{K}_t^c(x,y)$ as $\tilde{K}_0^c(x,y) = 1 + u_t(x,y)$ where $u_t(x,y)$ solves (4.47), namely (4.49). Finally, for $\mu \in [0, \sqrt{c/a})$, $\tilde{K}_0^c(x,y)$ is no longer bounded, only of exponential growth. Since we are interested in comparing to CARW and ARWI limits, we focus on the case $\mu > \sqrt{c/a}$. To find the limit of $\epsilon^{-1}\tilde{\mathbf{K}}^{(\epsilon)}(x,y)$, note that $1/\phi^{(\epsilon)} = 1 - \epsilon\sqrt{c/a} + O(\epsilon^2)$

and that $\epsilon^{-1}\tilde{\mathbf{K}}_{12}^{(\epsilon)}(x,x)$ converges as follows

$$\epsilon^{-1}\tilde{\mathbf{K}}_{12}^{(\epsilon)}(x,x) = -\frac{1}{\phi^{(\epsilon)}} D_{\epsilon}^{(0,1)} \tilde{K}_t^{(\epsilon)}(x,x) + \frac{1}{\epsilon}\left(1 - \frac{1}{\phi^{(\epsilon)}}\right)$$
$$\to -\partial_2 \tilde{K}_t^c(x,x) + \sqrt{\frac{c}{a}}.$$

We arrive at the following conjecture.

Conjecture 2. *Fix $t > 0$, $a > 0$, $b \in \mathbb{R}$, $c \geq 0$ and $\lambda_\epsilon \in [0,1]$ for $\epsilon > 0$. Suppose that $\epsilon^{-1}\lambda_\epsilon \to \mu$ with for some $\mu > \sqrt{c/a}$. Then the scaled BCRW point process $X^{(\epsilon)}$ with rates (4.54) and initial condition independent Bernoulli(λ_ϵ) converges in distribution to X^c, the Pfaffian point process on $\mathbb{R}$ with kernel $\tilde{\mathbf{K}}^c(x,y)$, given, for $x < y$, by*

$$\tilde{\mathbf{K}}^c(x,y) = -\begin{pmatrix} \tilde{K}_t^c(x,y) & \partial_2 \tilde{K}_t^c(x,y) \\ \partial_1 \tilde{K}_t^c(x,y) & \partial_1\partial_2 \tilde{K}_t^c(x,y) \end{pmatrix},$$

and $\tilde{\mathbf{K}}_{12}^c(x,x) = -\partial_2 \tilde{K}_t^c(x,x) + \sqrt{\frac{c}{a}}$. The function $(\tilde{K}_t^c(x,y) : t \geq 0,\, x,y \in \mathbb{R},\, x < y)$ is the unique bounded solution to (4.57). In particular, $\tilde{K}_t^c(x,y)$ has the explicit expression (4.44) with μ replaced by $\frac{1}{2}\left(\mu - \sqrt{\frac{c}{a}}\right) > 0$.

Remark 20. There is nothing special about the choice of rates (4.54) and the point process X^c will emerge as the scaling limit provided that the one-particle generators $\tilde{L}^{(\epsilon)}$ converge to L^c, namely that the coefficient satisfy (4.56). We examine the appropriate scaling of rates. Since BCRW reduces to CRW in the case $c = 0$, the jump rates should be weakly asymmetric. Consider the scaling of $r^{(\epsilon)}$. To satisfy condition (4.56) we must have

$$\frac{q^{(\epsilon)} + p^{(\epsilon)}}{2\epsilon^2}\left(1 - \phi^{(\epsilon)}\right)^2 \to c.$$

Note that $1/p^{(\epsilon)} = O(1)$ so $\phi^{(\epsilon)} = (1 + r^{(\epsilon)}O(1))^{1/2}$ and $r^{(\epsilon)}$ should scale as a positive power of ϵ to balance the ϵ^{-2} factor. This is intuitive: we must dampen the rate so that branching does not swamp the limit. We perform an order analysis to obtain the appropriate scaling for $r^{(\epsilon)}$. Note that $q^{(\epsilon)} + p^{(\epsilon)} = O(1)$ and by a Taylor expansion

$$\phi^{(\epsilon)} = \left(1 + r^{(\epsilon)}O(1)\right)^{\frac{1}{2}} = 1 + r^{(\epsilon)}O(1) + \left(r^{(\epsilon)}\right)^2 O(1).$$

All together, we obtain

$$\frac{q^{(\epsilon)} + p^{(\epsilon)}}{2\epsilon^2}\left(1 - \phi^{(\epsilon)}\right)^2 = \frac{O(1)}{\epsilon^2}\left(r^{(\epsilon)}\right)^2.$$

This expression must be $O(1)$, so the leading order of $r^{(\epsilon)}$ is ϵ. For the other branching rate $\ell^{(\epsilon)}$, note that $q^{(\epsilon)}/p^{(\epsilon)} = 1 + O(\epsilon)$ then condition (3.29a) gives $\ell^{(\epsilon)} = r^{(\epsilon)}q^{(\epsilon)}/p^{(\epsilon)} = r^{(\epsilon)}(1 + O(\epsilon))$. We see that $\ell^{(\epsilon)}$ also has leading order ϵ and moreover the branching rates are equal at this order. All together, we take equal branching rates at leading order ϵ and modify one of them at higher orders to satisfy condition (3.29a). We now see that condition (3.29a) does not represent genuine asymmetry, rather it ensures that the discrete point processes are Pfaffian and in doing so disguises the underlying symmetry in the branching rates. This now raises the question of whether we lose generality via the modified empty interval probability procedure. However, the impression of there being a different class of continuum models with asymmetric branching is illusory, as we now demonstrate.

Consider the BCRW model on $\mathbb{Z}$ with constant jump rates p, q, and constant branching rates r, ℓ. By lemma 5 and rewriting (3.33) and (3.34), the empty interval probability $K_t(x, y)$, defined in (3.28), satisfies the differential equation

$$\partial_t K_t(x, y) = (L_x^{(1)} + L_y^{(2)})K_t(x, y),$$

where the one-particle generators are given, for $f : \mathbb{Z} \to \mathbb{R}$, by

$$\begin{aligned} L^{(1)}f(x) &= \frac{q + p + r}{2}\Delta_x^{(1)}f(x) + (q - p - r)\nabla_x^{(1)}f(x), \\ L^{(2)}f(x) &= \frac{q + p + \ell}{2}\Delta_x^{(1)}f(x) + (q - p + \ell)\nabla_x^{(1)}f(x). \end{aligned}$$

Fixing rates $p^{(\epsilon)}$, $q^{(\epsilon)}$, $r^{(\epsilon)}$ and $\ell^{(\epsilon)}$, the empty interval probability $K_t^{(\epsilon)}$ for the scaled process (4.35) satisfies the differential equation with the following one-particle generators for $f : \epsilon\mathbb{Z} \to \mathbb{R}$

$$\begin{aligned} L^{(1),(\epsilon)}f(x) &= \frac{q^{(\epsilon)} + p^{(\epsilon)} + r^{(\epsilon)}}{2}\Delta_x^{(\epsilon)}f(x) + \epsilon^{-1}\big(q^{(\epsilon)} - p^{(\epsilon)} - r^{(\epsilon)}\big)\nabla_x^{(\epsilon)}f(x), \\ L^{(2),(\epsilon)}f(x) &= \frac{q^{(\epsilon)} + p^{(\epsilon)} + \ell^{(\epsilon)}}{2}\Delta_x^{(\epsilon)}f(x) + \epsilon^{-1}\big(q^{(\epsilon)} - p^{(\epsilon)} + \ell^{(\epsilon)}\big)\nabla_x^{(\epsilon)}f(x). \end{aligned}$$

We pick weakly asymmetric jump rates and order ϵ branching rates satisfying

$$\frac{p^{(\epsilon)} + q^{(\epsilon)}}{2} \to a, \qquad \frac{q^{(\epsilon)} - p^{(\epsilon)}}{\epsilon} \to b, \qquad \epsilon^{-1}r^{(\epsilon)} \to r_0, \qquad \epsilon^{-1}\ell^{(\epsilon)} \to \ell_0, \tag{4.58}$$

as $\epsilon \downarrow 0$ for some $a > 0$, $b \in \mathbb{R}$ and $r_0, \ell_0 \geq 0$. The continuum limit K_t^c of $K_t^{(\epsilon)}$ satisfies

$$\partial_t K_t^c(x,y) = a\Delta K_t^c(x,y) + (b - r_0)\partial_x K_t^c(x,y) + (b + \ell_0)\partial_y K_t^c(x,y).$$

This may be converted into the differential equation for equal branching rates by a suitable Galilean transformation, corresponding to adding a global drift to the system. The change of variables is given by

$$t' = t, \qquad x' = x - vt, \qquad y' = y - vt, \qquad \text{for some } v \in \mathbb{R},$$

and in the transformed system we have

$$\partial_{t'} K_{t'}^c(x',y') = a\Delta K_{t'}^c(x',y') + (v + b - r_0)\partial_{x'} K_{t'}^c(x',y') + (v + b + \ell_0)\partial_{y'} K_{t'}^c(x',y').$$

Choosing $v = \frac{r_0 - \ell_0}{2} - b$ the differential equation becomes

$$\partial_{t'} K_{t'}^c(x',y') = a\Delta K_{t'}^c(x',y') - \beta\partial_{x'} K_{t'}^c(x',y') + \beta\partial_{y'} K_{t'}^c(x',y'),$$

where $\beta = \frac{r_0 + \ell_0}{2} \geq 0$. This corresponds to the limit of scaled processes with symmetric jump rates and symmetric branching rates, that is, with $b = 0$ and $r_0 = \ell_0 = \beta$ in (4.58). We already know that jump rate asymmetry translates into a drift b. The above shows that the manifestation of branching rate asymmetry is also an effective drift. Thus, for constant rates, the family of symmetric branching rate continuum point processes contains all BCRW scaling limits. For independent Bernoulli initial conditions this class is characterised by conjecture 2. In other words, developing BCRW via modified empty interval probabilities does not embody a loss of generality.

4.4.2 Continuum relation between branching and immigration

A relation between the continuum scaling limits of ARWI and BCRW, under particular initial conditions, is derived. At the heart of the result is theorem 4 relating the underlying ARWI and BCRW models. Unlike the discrete case, the superposition of continuum point processes that do not charge given singletons is unambiguous and the relation follows quickly from the kernels.

Proposition 19. *Fix $a > 0$, $c \geq 0$ and $\mu > \sqrt{\frac{c}{a}}$. Consider the following independent point processes on $\mathbb{R}$:*

- *X_μ^A, the Pfaffian scaling limit of ARWI in conjecture 1;*

- *X^B_μ, the Pfaffian scaling limit of BCRW in conjecture 2;*
- *Y, a rate $\frac{1}{2}\sqrt{\frac{c}{a}}$ Poisson process.*

Then the law of the process X^B_μ thinned at rate $\frac{1}{2}$ is equal to the law of the superposition of $X^A_{\mu'}$ and Y, where $\mu' = \frac{1}{2}\left(\mu - \sqrt{\frac{c}{a}}\right)$.

Proof of proposition 19. The kernel $\mathbf{K}^A_\mu$ for X^A_μ is given by (4.42) and (4.43). The kernel for X^B_μ is $2\mathbf{K}^A_{\mu'}(x,y) + \sqrt{\frac{c}{a}}J1(x=y)$ where $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Proposition 15 implies that the point process X^B_μ thinned at rate $\frac{1}{2}$ is Pfaffian with kernel $\mathbf{K}^A_{\mu'}(x,y) + \frac{1}{2}\sqrt{\frac{c}{a}}J1(x=y)$. On the other hand, the superposition of $X^A_{\mu'}$ and Y also has this kernel by proposition 18. Finally, since the kernel is uniformly bounded, the law of the associated Pfaffian point process is determined by proposition 13. □

Consulting the comments proceeding theorem 6, we expect that when $c = 0$ the point processes X^A_μ and X^B_μ correspond to systems of annihilating and coalescing Brownian motions, respectively, with Poisson initial conditions. Proposition 19 for $c = 0$ then gives the associated thinning relation (see remark 3). Indeed note that in this case Y is the empty point process and $\mu' = \mu/2$. When $\mu = \infty$ we recover the thinning relation between coalescing and annihilating Brownian motions under the maximal entrance law.

Repeating the ARWI steps of section 4.3.2 and using the thickening result proposition 18, the invariant distribution for the BCRW limit is a rate $2\sqrt{\frac{c}{a}}$ Poisson process. This is consistent with the above relation, which in the case $t = \infty$ is simply the algebra of independent Poisson processes.

Chapter 5

Extended Pfaffian point processes

We have so far only described the one-dimensional marginals of interacting particle systems as Pfaffian. We now develop a generalised construction of Pfaffian point processes on multiple copies of $\mathbb{Z}$ or $\mathbb{R}$, the so-called extended property, enabling a multi-time Pfaffian description of interacting particle systems. The ARWI model of section 3.3.1 is shown to be an extended Pfaffian point process.

An introduction to extended Pfaffian point processes is given in section 5.1. We prove in section 5.2 the extended Pfaffian property for ARWI and compare with annihilating Brownian motions.

5.1 Definition and examples

We introduce extended Pfaffian point processes, giving definitions as well as some known examples.

Our random walk interacting particle systems are time-homogeneous continuous-time Markov processes taking values in the set of simple measures $\mathcal{M}_0(\mathbb{R})$. We have so far described the one-dimensional marginals and their diffusive scaling limits. Our investigation now turns to characterising the evolution of processes via multi-time statistics. The concept of intensity function extends to time-dependent point processes, taking a set of space-time points as its argument and, at least informally, returning the probability of the process occupying those positions. The multi-time intensity functions determine the finite-dimensional distributions for the stochastic

process evolving in $\mathcal{M}_0(\mathbb{R})$. Accordingly, one hopes that, given certain regularity of the measures, the multi-time intensities characterise the process.

The structure of Pfaffians may be incorporated by asking for a multi-time intensity to be given by the Pfaffian of a matrix indexed by space-time points and generated by a kernel $\mathbf{K}(s, x; t, y)$ taking two space-time arguments. This is the essence of an extended Pfaffian point process. Intuitively a time-dependent process is an extended Pfaffian point process on Λ if the finite-dimensional marginal for fixed times $t_1 < \cdots < t_L$ is a Pfaffian point process on the state space $\Lambda \amalg \ldots \amalg \Lambda$ (L copies), where $\amalg$ denotes the disjoint union of sets. Note that by taking $L = 1$ we recover the Pfaffian structure for single times.

We will show in section 5.2 that ARWI is an extended Pfaffian point process on $\mathbb{Z}$. We do not develop here the convergence theory for extended processes and, as such, we focus on the definition for discrete processes. In this case, the multi-time intensity is exactly the aforementioned occupation probability. Let $(X_t : t \geq 0)$ be a collection of simple point processes on $\mathbb{Z}$, for us, a continuous-time Markov process on $\{0, 1\}^{\mathbb{Z}}$. For $n \in \mathbb{N}$, $t_1, \ldots, t_n \in [0, \infty)$ and $x_1, \ldots, x_n \in \mathbb{Z}$ satisfying $(t_i, x_i) \neq (t_j, x_j)$ for $i \neq j$, the multi-time intensity is defined by

$$\rho^{(n)}(t_1, x_1; \ldots; t_n, x_n) = \mathbb{E}\left[X_{t_1}(x_1) \ldots X_{t_n}(x_n)\right].$$

An *extended kernel* is defined to be a 2×2 matrix-valued function $\mathbf{K} : ([0, \infty)\times\mathbb{Z})^2 \to \mathbb{R}^{2\times 2}$ with $\mathbf{K}(s, x; t, y) = \begin{pmatrix} \mathbf{K}_{11}(s,x;t,y) & \mathbf{K}_{12}(s,x;t,y) \\ \mathbf{K}_{21}(s,x;t,y) & \mathbf{K}_{22}(s,x;t,y) \end{pmatrix}$, constructed by defining

$$\mathbf{K}_{ij} : ([0, \infty) \times \mathbb{Z})^2 \to \mathbb{R},$$

satisfying the anti-symmetry condition $\mathbf{K}_{ij}(s, x; t, y) = -\mathbf{K}_{ji}(t, y; s, x)$.

Definition 8. *Let $(X_t : t \geq 0)$ be a collection of simple point processes on $\mathbb{Z}$ with multi-time intensities $\{\rho^{(n)} : n \in \mathbb{N}\}$. Suppose that there exists an extended kernel $\mathbf{K}$ such that*

$$\rho^{(n)}(t_1, x_1; \ldots; t_n, x_n) = \mathrm{Pf}\left(\mathbf{K}(t_i, x_i; t_j, x_j) : i, j \leq n\right), \tag{5.1}$$

*for $t_1, \ldots, t_n \in [0, \infty)$ and $x_1, \ldots, x_n \in \mathbb{Z}$. Then $(X_t : t \geq 0)$ is called an **extended Pfaffian point process** on $\mathbb{Z}$ with (extended) kernel $\mathbf{K}$, and is said to satisfy the **extended Pfaffian property**.*

The notation $\mathrm{Pf}\left(\mathbf{K}(t_i, x_i; t_j, x_j) : i, j \leq n\right)$ represents the Pfaffian of the $2n \times 2n$

matrix generated by the n^2 kernel blocks $\mathbf{K}(t_i, x_i; t_j, x_j)$ where the arguments range over $(t_1, x_1), \ldots, (t_n, x_n)$. For fixed L, the space $\Lambda = \mathbb{Z} \amalg \ldots \amalg \mathbb{Z}$ (L copies) is a locally compact, second countable Hausdorff space, in the disjoint union topology, and so the characterisation of the extended property as finite-dimensional marginals being Pfaffian on Λ is well-defined. Through this interpretation, extended Pfaffian point processes inherit several properties immediately, for instance, the multi-time intensity vanishes if $(t_i, x_i) = (t_j, x_j)$ for some $i \neq j$. Moreover the ordering of points makes no difference and for convenience we generally order by temporal and then spatial position. Proposition 12 applies to extended processes, giving equivalent extended kernels. Invoking proposition 13 guarantees a suitably bounded extended kernel determines the law of the finite-dimensional marginals of $(X_t : t \geq 0)$ on $\mathbb{Z} \amalg \ldots \amalg \mathbb{Z}$. We expect stronger results to hold for any extended Pfaffian random walk model, since there is an underlying continuous-time Markov process.

Extended Pfaffian point processes on $\mathbb{R}$ are defined similarly: multi-time intensity functions are introduced and must satisfy condition (5.1) for $x_1, \ldots, x_n \in \mathbb{R}$.

Examples of extended Pfaffian point processes are still fairly uncommon. The key example is a system of annihilating Brownian motions.

Example 10 (Annihilating Brownian motions on $\mathbb{R}$)**.** In [58] Tribe and Zaboronski showed that annihilating Brownian motions under a maximal entrance law are an extended Pfaffian point process on $\mathbb{R}$. This extends their result of [59] for the one-dimensional distributions, see example 6. The extended kernel is given by convolving the single-time kernel $\tilde{\mathbf{K}}_t^{\mathrm{ABM}}$, defined by (2.13), and the heat kernel $p_t^c(z) = (2\pi t)^{-1/2} e^{-z^2/2t}$, namely for $s < t$ and $x, y \in \mathbb{R}$

$$\mathbf{K}_{ij}^{\mathrm{ABM}}(s, x; t, y) = -\left(p_{t-s}^c \star (\tilde{\mathbf{K}}_s^{\mathrm{ABM}})_{ij}\right)(x, y) + 1(i = 2, j = 1) p_{t-s}^c(y - x). \quad (5.2)$$

Note that $\tilde{\mathbf{K}}_t^{\mathrm{ABM}}(x, y)$ depends on $y - x$ and so the one-dimensional convolution makes sense. The additional term in the $(2, 1)$ entry represents the event that the particle at (t, y) evolved from (s, x). (Note that the kernel in [58] contains a typo: the additional term should have a prefactor of -1 instead of -2.)

Example 11 (A Markov process on strict partitions)**.** In [42] Petrov introduced a continuous time Markov process on the space of strict partitions $\lambda = (\lambda_1 > \cdots > \lambda_L > 0)$ where $\lambda_i \in \{1, 2, \ldots\}$. This space may be identified with certain types of Young diagrams. The one-dimensional marginals coincide with a determinantal random strict partition model of Borodin. The dynamical model $(\lambda_t : t \geq 0)$ may be

defined by the transition probabilities for adding and deleting boxes from the Young diagrams, which depend on a continuous time birth-death process. It is shown that $(\lambda_t : t \geq 0)$ is an extended Pfaffian point process with kernel given in terms of the Gauss hypergeometric function. See [42] for a full description.

As for single times (example 3), there is a determinantal counterpart to extended processes. An *extended determinantal point process* is defined by replacing the Pfaffian in (5.1) by a determinant and the kernel by $\mathbf{K} : ([0,\infty)\times\mathbb{R})^2 \to \mathbb{R}$. Extended determinantal point process have received more attention than Pfaffian due to explicit tractable examples arising in random matrix theory and interacting particle systems. In particular, extended determinantal point processes arise when the multi-time intensities are given by a product of determinants [31], which arise naturally in systems of non-interacting Markov processes due to the determinantal Karlin-McGregor transition probability [34] and its discrete analogue, the Lindström-Gessel-Viennot theorem [54]. We remark that the epithet 'extended' is normally suppressed, since the context is clear from the form of the kernel.

Example 12 (Dyson's Brownian motion)**.** The GUE ($\beta = 2$) model of random matrix theory is introduced in example 5. The ensemble comprises $N \times N$ Hermitian matrices with independent complex Gaussian entries. Dyson [16] considered a temporal extension in which the entries evolve as independent Brownian motions. More precisely, define a Hermitian matrix-valued Brownian motion $\Big(H^{(N)}(t) = \big(H_{ij}^{(N)}(t)\big)_{i,j=1}^{N} : t \geq 0\Big)$ by its entries

$$H_{ij}^{(N)}(t) = \begin{cases} \frac{1}{\sqrt{N}}(B_{ij}(t) + iB'_{ij}(t)) & \text{for } i < j, \\ \frac{1}{\sqrt{N}}(B_{ij}(t) - iB'_{ij}(t)) & \text{for } i > j, \\ \sqrt{\frac{2}{N}}B_{ii}(t) & \text{for } i = j, \end{cases}$$

where B_{ij}, B'_{ij} are independent standard Brownian motions. Dyson showed that the eigenvalues $\lambda_1^{(N)}(t) < \lambda_2^{(N)}(t) < \cdots < \lambda_N^{(N)}(t)$ solve the following system of stochastic differential equations

$$\mathrm{d}\lambda_i^{(N)}(t) = \frac{1}{\sqrt{N}}\mathrm{d}W_i(t) + \frac{1}{N}\sum_{j\neq i}\frac{1}{\lambda_i^{(N)}(t) - \lambda_j^{(N)}(t)}\mathrm{d}t, \qquad \text{for } i = 1,\ldots,N,$$

where W_i are independent standard Brownian motions. This is known as *Dyson's Brownian motion*. The eigenvalues evolve as Brownian motions with an additional repulsion term guaranteeing they stay ordered. In other words, the eigenvalues

form a system of non-colliding Brownian motions. The intensities are given by determinants of the *extended Hermite kernel* [19, 30], so that Dyson's Brownian motion is an extended determinantal point process for each N.

We now consider scaling limits, referring to [1], for instance, and references therein for further details and formulae. Scaling the extended Hermite kernel leads to two more extended kernels, describing limiting behaviour in the bulk or near the edge of Dyson's Brownian motion as $N \to \infty$. Scaling in the bulk gives the *Sine process* [57] defined by the *extended sine kernel*. At the edge, scaling near the largest eigenvalue $\lambda_N^{(N)}$ leads to the *Airy process* [43] defined by the *extended Airy kernel*. Both extended determinantal point processes correspond to infinite collections of non-colliding processes. Upon taking coincidental times, the extended sine and extended Airy kernels reduce to the ordinary sine and Airy kernels for the GUE scaling limits (see example 5).

Generalising example 3, an extended determinantal point processes with kernel $K(s,x;t,y)$ is trivially Pfaffian, for example take the extended kernel $\mathbf{K}(s,x;t,y) = \begin{pmatrix} 0 & K(s,x;t,y) \\ -K(t,y;s,x) & 0 \end{pmatrix}$.

5.2 Extended Pfaffian property for annihilating random walks with pairwise immigration

The ARWI model of section 3.3.1 is shown to be an extended Pfaffian point process for a wide class of initial conditions, including deterministic.

In [58], a system of annihilating Brownian motions, under a maximal entrance law, was shown to be an extended Pfaffian point process on $\mathbb{R}$. This builds on the work of [59], which established the Pfaffian property at single times. Sections 3.2 and 3.3.1 extended these single-time results to ARWI and the aim of this section is to do the same for the multi-time results of [58]. Precisely, we extend the annihilating Brownian motion result by considering: (i) analogous particle systems on $\mathbb{Z}$ with an (optional) additional mechanism of pairwise immigration; (ii) spatially inhomogeneous nearest neighbour motion; (iii) general deterministic initial conditions. The extended Pfaffian structure survives all of these changes.

By theorem 2 ARWI at a fixed time $t \geq 0$ is Pfaffian with kernel $\mathbf{K}(x,y)$ given by (3.20), in terms of the spin expectation $K_t(x,y)$ defined on $\{x \leq y\} \subset \mathbb{Z}^2$. The

extended kernel must be defined for space-time points (s, x) and (t, y) in $[0, \infty) \times \mathbb{Z}$. Anti-symmetry allows us to just define the kernel at times $s \leq t$, but we must consider spatial points x, $y \in \mathbb{Z}$. Because of this, it is convenient to extend $K_t(x, y)$ to $\mathbb{Z}^2$. We define $\tilde{K}_t : \mathbb{Z}^2 \to \mathbb{R}$ by

$$\tilde{K}_t(x, y) = K_t(\min\{x, y\}, \max\{x, y\}) \operatorname{sgn}(y - x).$$

Note that $\tilde{K}_t(x, x) = 0 \neq 1 = K_t(x, x)$, however $\mathbf{K}(x, y)$ may be written in terms of $\tilde{K}_t(x, y)$ at the expense of additional indicator terms. The single-time parameter plays a role in the extended picture and we append t to the subscript of the kernel. The single-time kernel $\mathbf{K}_t(x, y)$ for ARWI at time t is given, for $x, y \in \mathbb{Z}$, by

$$\begin{aligned}
(\mathbf{K}_t)_{11}(x, y) &= -\tfrac{1}{2}\tilde{K}_t(x, y), \\
(\mathbf{K}_t)_{12}(x, y) &= -\tfrac{1}{2}\left(D_2^+ \tilde{K}_t(x, y) - 1(x = y + 1) - 1(x = y)\right), \\
(\mathbf{K}_t)_{21}(x, y) &= -\tfrac{1}{2}\left(D_1^+ \tilde{K}_t(x, y) + 1(x = y - 1) + 1(x = y)\right), \\
(\mathbf{K}_t)_{22}(x, y) &= -\tfrac{1}{2}\left(D_1^+ D_2^+ \tilde{K}_t(x, y) + 1(x = y + 1) - 1(x = y - 1)\right).
\end{aligned} \tag{5.3}$$

The extended kernel for ARWI is built from the one-dimensional convolution, and its derivatives, of $\tilde{K}_t$ against the Green's function $p_t : \mathbb{Z}^2 \to \mathbb{R}$ for the one-particle generator L^A, given by (3.21), on $\mathbb{Z}$, namely $p_t(x, y)$ is the solution to

$$\begin{cases} \partial_t p_t(x, y) = L_y^A p_t(x, y) & \text{for } y \in \mathbb{Z},\ t > 0, \\ p_0(x, y) = 1(x = y) & \text{for } y \in \mathbb{Z}. \end{cases}$$

For a function $f : \mathbb{Z}^2 \to \mathbb{R}$, one-dimensional convolution with p_t is defined by

$$(p_t * f)(x, y) = \sum_{z \in \mathbb{Z}} p_t(z, y) f(x, z), \qquad \text{for } x,\ y \in \mathbb{Z}.$$

We can now state the main result of this section.

Theorem 7. *For any initial condition, $\eta_0 \in \{0, 1\}^{\mathbb{Z}}$, the ARWI system $(\eta_t : t \geq 0)$ is an extended Pfaffian point process on $\mathbb{Z}$ with kernel $\mathbf{K}(s, x; t, y)$ given, for $s < t$*

and $i, j \in \{1, 2\}$*, by*

$$\begin{aligned}
\mathbf{K}_{11}(s, x; t, y) &= -\tfrac{1}{2}\Big(\Big(p_{t-s} * \tilde{K}_s\Big)(x, y) - p_{t-s}(x, y)\Big), \\
\mathbf{K}_{12}(s, x; t, y) &= -\tfrac{1}{2}\Big(D_2^+\Big(p_{t-s} * \tilde{K}_s\Big)(x, y) - p_{t-s}(x, y+1) + p_{t-s}(x, y)\Big), \\
\mathbf{K}_{21}(s, x; t, y) &= -\tfrac{1}{2}\Big(D_1^+\Big(p_{t-s} * \tilde{K}_s\Big)(x, y) + p_{t-s}(x, y-1) + p_{t-s}(x, y)\Big), \\
\mathbf{K}_{22}(s, x; t, y) &= -\tfrac{1}{2}\Big(D_1^+ D_2^+\Big(p_{t-s} * \tilde{K}_s\Big)(x, y) + p_{t-s}(x, y+1) - p_{t-s}(x, y-1)\Big).
\end{aligned} \tag{5.4}$$

and, for $s = t$*, by*

$$\mathbf{K}(s, x; t, y) = \mathbf{K}_t(x, y). \tag{5.5}$$

The $s > t$ *entries are determined by anti-symmetry of the kernel.*

Remark 21. The derivatives in the extended kernel (5.4) may be moved inside the convolution

$$\begin{aligned}
\mathbf{K}_{11}(s, x; t, y) &= -\tfrac{1}{2}\Big(\Big(p_{t-s} * \tilde{K}_s\Big)(x, y) - p_{t-s}(x, y)\Big), \\
\mathbf{K}_{12}(s, x; t, y) &= -\tfrac{1}{2}\Big(\Big(\big(D_2^+ p_{t-s}\big) * \tilde{K}_s\Big)(x, y) - p_{t-s}(x, y+1) + p_{t-s}(x, y)\Big), \\
\mathbf{K}_{21}(s, x; t, y) &= -\tfrac{1}{2}\Big(\Big(p_{t-s} * \big(D_1^+ \tilde{K}_s\big)\Big)(x, y) + p_{t-s}(x, y-1) + p_{t-s}(x, y)\Big), \\
\mathbf{K}_{22}(s, x; t, y) &= -\tfrac{1}{2}\Big(\Big(\big(D_2^+ p_{t-s}\big) * \big(D_1^+ \tilde{K}_s\big)\Big)(x, y) + p_{t-s}(x, y+1) - p_{t-s}(x, y-1)\Big).
\end{aligned}$$

In the case of constant rates, $p_x = p$, $q_x = q$ and $m_x = m$ for all $x \in \mathbb{Z}$, the function p_{t-s} is spatially invariant, depending on x and y only through $y - x$. In this case the derivatives in the convolution may be moved between functions and the extended kernel (5.4) can be expressed as a convolution of the single-time kernel with two additional terms

$$\mathbf{K}_{ij}(s, x; t, y) = \left(p_{t-s} * (\mathbf{K}_s)_{ij}\right)(x, y) + \left(\frac{1}{2} 1(i = j = 1) - 1(i = 1, j = 2)\right) p_{t-s}(x, y). \tag{5.6}$$

For example, writing the spatially homogeneous Green's function as $p_{t-s}(y - x)$ for

convenience, the $(1,2)$ entry of $\mathbf{K}(s,x;t,y)$ is given by

$$\begin{aligned}
&(-2)\mathbf{K}_{12}(s,x;t,y) \\
&\quad = \sum_{z\in\mathbb{Z}} \left(p_{t-s}(y+1-z) - p_{t-s}(y-z)\right)\tilde{K}_s(x,z) - p_{t-s}(y+1-x) + p_{t-s}(y-x) \\
&\quad = \sum_{z\in\mathbb{Z}} p_{t-s}(y-z)\left(\tilde{K}_s(x,z+1) - \tilde{K}_s(x,z)\right) - p_{t-s}(y+1-x) + p_{t-s}(y-x) \\
&\quad = \sum_{z\in\mathbb{Z}} p_{t-s}(y-z)\left(D_2^+\tilde{K}_s(x,z) - 1(x=z-1) - 1(x=z)\right) + 2p_{t-s}(y-x).
\end{aligned}$$

The claimed expression then follows from (5.3). The other entries may be developed similarly. We develop the case of symmetric annihilating random walks under the maximal initial condition in remark 27 after the proof, comparing with the annihilating Brownian motion model of example 10.

Remark 22. For general rates the Green's function is spatially invariant in the limit $\lim_{r\downarrow 0} p_r(x,y) = 1(x=y)$. Thus, the limit as $t \downarrow s$ in the extended kernel (5.4) coincides with the limit of (5.6), namely the single-time kernel with indicator terms in the $\mathbf{K}_{11}$ and $\mathbf{K}_{12}$ entries

$$\lim_{t\downarrow s} \mathbf{K}_{ij}(s,x;t,y) = (\mathbf{K}_s)_{ij}(x,y) + \left(\frac{1}{2}1(i=j=1) - 1(i=1,j=2)\right)1(x=y).$$

The additional indicator terms arise due to the event that the particle at (t,y) evolved from the particle at (s,x) (instead of the two space-time points representing different particles). The necessity for additional terms can be seen through the following consistency check (based on the heuristic remark in [58]) for the one-point intensity $\mathbb{E}\left[\eta_s(x)\right] = -\frac{1}{2}D_2^+K_s(x,x)$. For $x \in \mathbb{Z}$

$$\mathbb{E}\left[\eta_s(x)\right] = \mathbb{E}\left[\eta_s(x)^2\right] = \lim_{t\downarrow s}\mathbb{E}\left[\eta_s(x)\eta_t(x)\right],$$

and by theorem 7 the multi-time intensity is given by the Pfaffian

$$\mathrm{Pf}\begin{pmatrix}
0 & (\mathbf{K}_s)_{12}(x,x) & \mathbf{K}_{11}(s,x;t,x) & \mathbf{K}_{12}(s,x;t,x) \\
-(\mathbf{K}_s)_{12}(x,x) & 0 & \mathbf{K}_{21}(s,x;t,x) & \mathbf{K}_{22}(s,x;t,x) \\
-\mathbf{K}_{11}(s,x;t,x) & -\mathbf{K}_{21}(s,x;t,x) & 0 & (\mathbf{K}_t)_{12}(x,x) \\
-\mathbf{K}_{12}(s,x;t,x) & -\mathbf{K}_{22}(s,x;t,x) & -(\mathbf{K}_t)_{12}(x,x) & 0
\end{pmatrix}.$$

Expanding the Pfaffian, taking limits and substituting in the single-time kernel

(3.20) gives the required intensity

$$\begin{aligned}\lim_{t\downarrow s}\mathbb{E}\left[\eta_s(x)\eta_t(x)\right] &= \left((\mathbf{K}_s)_{12}(x,x)\right)^2 - \left((\mathbf{K}_s)_{11}(x,x)+\frac{1}{2}\right)(\mathbf{K}_s)_{22}(x,x)\\ &\quad + \left((\mathbf{K}_s)_{12}(x,x)-1\right)(\mathbf{K}_s)_{21}(x,x)\\ &= \frac{1}{4}\left(D_2^+K_s(x,x)\right)^2 - \left(-\frac{1}{2}K_s(x,x)+\frac{1}{2}\right)\left(-\frac{1}{2}D_1^+D_2^+K_s(x,x)\right)\\ &\quad + \left(-\frac{1}{2}D_2^+K_s(x,x)-1\right)\frac{1}{2}D_2^+K_s(x,x)\\ &= -\frac{1}{2}D_2^+K_s(x,x).\end{aligned}$$

Without the additional indicator terms the limit is given by $-\frac{1}{4}D_1^+D_2^+K_s(x,x)$ and we do not recover the intensity. The extended kernel for the annihilating Brownian motion system in [58] has the same form of a convolution between the single-time kernel and the Green's function, with the $(1,2)$ entry featuring an additional Green's function term with prefactor -1. There is no additional Green's function term in the $(1,1)$ entry; this is purely a discrete artefact. Indeed for suitable approximating ARW systems the term disappears in the diffusive scaling limit and we recover the annihilating Brownian motion extended kernel (see remark 27 after the proof of theorem 7).

Theorem 7 is proved by showing that the multi-time intensities satisfy (5.1) with the claimed kernel. The proof is based on the method in [58] for annihilating Brownian motions. The idea is to extend the ODE characterisation result for spin expectations, lemma 4, to certain multi-time mixed expectations of intensities and spins. A Pfaffian representation for these expectations may be established inductively and theorem 7 is recovered as a special case. An induction is performed on the number of disjoint times, so it is convenient to partition and relabel the space-time points according to the temporal component. We prove the following result.

Theorem 8. *Fix a number of time slices $L \in \mathbb{N}$ and spin pairs $m \in \mathbb{N}$. For times $t_1 < \cdots < t_L \leq t$, particle counts $n_1, \ldots, n_L$, particle positions $x_1^{(\ell)} < \cdots < x_{n_\ell}^{(\ell)}$ in $\mathbb{Z}$ at time t_ℓ for $1 \leq \ell \leq L$, and spin positions $y_1 \leq \cdots \leq y_{2m}$, the following Pfaffian*

expression holds for a mixed intensity and spin expectation

$$\mathbb{E}\left[\prod_{\ell=1}^{L}\prod_{i=1}^{n_\ell}\eta_{t_\ell}\left(x_i^{(\ell)}\right)\prod_{j=1}^{m}(-1)^{\eta_t[y_{2j-1},y_{2j})}\right] = (-2)^{-\sum_{\ell=1}^{L}n_\ell}\operatorname{Pf}\left(\tilde{\mathbf{K}}(s_1,w_1;s_2,w_2):(s_1,w_1),\,(s_2,w_2)\in A\right), \quad (5.7)$$

where the extended kernel $\tilde{\mathbf{K}}$ *ranges over space-time points from*

$$A = \underbrace{\bigcup_{\ell=1}^{L}\bigcup_{i=1}^{n_\ell}\left(t_\ell, x_i^{(\ell)}\right)}_{Particles} \cup \underbrace{\bigcup_{j=1}^{2m}(t,y_j)}_{Spins}, \quad (5.8)$$

with the particle positions $(t_\ell, x_i^{(\ell)})$ *appearing before the (ordered) spin positions* (t, y_j)*, and is defined as follows: for* $t_\ell \le t_k$*,* $x, w \in \mathbb{Z}$ *and* $i < j$

$$\begin{aligned}
\tilde{\mathbf{K}}(t_\ell,x;t_k,w) &= -2\mathbf{K}(t_\ell,x;t_k,w) && (2\times 2\ \textit{entry}),\\
\tilde{\mathbf{K}}(t_\ell,x;t,y_j) &= \\
&\begin{pmatrix} \left(p_{t-t_\ell} * \tilde{K}_{t_\ell}\right)(x,y_j) - p_{t-t_\ell}(x,y_j) \\ D_1^+\left(p_{t-t_\ell} * \tilde{K}_{t_\ell}\right)(x,y_j) + p_{t-t_\ell}(x,y_j-1) + p_{t-t_\ell}(x,y_j) \end{pmatrix} && (2\times 1\ \textit{entry}),\\
\tilde{\mathbf{K}}(t,y_i;t,y_j) &= K_t(y_i,y_j) && (1\times 1\ \textit{entry}).
\end{aligned} \quad (5.9)$$

The remaining entries are defined by anti-symmetry.

Note that theorem 7 is recovered upon setting $m = 0$, since the factors of $-1/2$ may be moved onto the kernel (proposition 7). We turn attention to the proof of theorem 8. For convenience we introduce the shorthand $\operatorname{Pf}(\tilde{\mathbf{K}}(A))$ for the Pfaffian on the right-hand side of (5.7).

Remark 23. Theorem 8, and hence theorem 7, hold for certain random initial conditions. Indeed by virtue of the inductive proof, we need only check the claim for spin expectations. This is already addressed in remark 6 for ARW, extending immediately to ARWI. The condition is that the spin expectations at time zero are Pfaffian. Independent Bernoulli initial conditions are such an example.

Remark 24. In (5.7) the ordering of points $(t_\ell, x_i^{(\ell)})$ in the matrix does not affect the Pfaffian and for convenience we fix the natural ordering via temporal, then spatial,

components. The assumption that the particle positions $(t_\ell, x_i^{(\ell)})$ appear before the (ordered) spin positions (t, y_j) means that (5.9) defines the upper triangular entries for $\mathrm{Pf}(\tilde{\mathbf{K}}(A))$. The general spin-spin entry for example is given by

$$\tilde{\mathbf{K}}(t, y_i; t, y_j) = K_t(\min\{y_i, y_j\}, \max\{y_i, y_j\}) \operatorname{sgn}(j - i).$$

Remark 25. A natural guess for how to extend the single-time result would be to consider multi-time spin expectations and then use the explicit connection (3.14) between spins and intensities to derive the multi-time intensities. Unfortunately, however, multi-time spin expectations are not Pfaffian. The mixed expectation acts as a suitable intermediary between multi-time spin expectations and multi-time intensities. In fact, by its inductive proof, theorem 8 shows that a suitable way to develop multi-time intensities is to build them up from spin expectations one time slice at a time.

Remark 26. Although we follow the continuum blueprint for annihilating Brownian motions, there are important differences on the lattice. As we have already seen for single times, switching between spins and intensities by (3.14) is straightforward for discrete processes, whereas the continuum analogue is given by a distributional derivative, forcing technical consideration. On the other hand, this explicit connection involves adjacent lattice sites, introducing subtleties with coincidental spin positions. Moreover the model we address here is spatially inhomogeneous and the formulae are accordingly more involved.

Proof of theorem 8. The proof proceeds via an induction on the number of time slices L. The base of the induction, $L = 0$, is the content of lemma 4. For $L > 0$ we follow a similar strategy of characterising the mixed expectations by a system of ODEs in the spin positions and showing that the claimed Pfaffians are also solutions. The inductive hypothesis is used to establish the boundary and initial conditions. Fix $L > 0$ and particle positions $x_1^{(\ell)} < \cdots < x_{n_\ell}^{(\ell)}$ for $1 \leq \ell \leq L$, then for $\mathbf{y} = (y_1, \ldots, y_{2m})$ with $y_1 \leq \cdots \leq y_{2m}$, set

$$u^{(2m)}(t, \mathbf{y}) = \mathbb{E}\left[\prod_{\ell=1}^{L}\prod_{i=1}^{n_\ell} \eta_{t_\ell}\left(x_i^{(\ell)}\right) \prod_{j=1}^{m} (-1)^{\eta_t[y_{2j-1}, y_{2j})}\right].$$

Using the notation and framework of lemma 2, $u^{(2m)}(t, \mathbf{y})$ solves the following system

of ODEs in the variables $t \in [t_L, \infty)$ and $\mathbf{y} \in V_{2m}$

$$(\mathrm{ODE})'_{2m} \quad \begin{cases} \partial_t u^{(2m)}(t,\mathbf{y}) &= \sum_{j=1}^{2m} L^A_{y_j} u^{(2m)}(t,\mathbf{y}) & \text{on } [t_L,\infty)\times V_{2m}, \\ u^{(2m)}(t,\mathbf{y}) &= u^{(2m-2)}(t,\mathbf{y}^{j,j+1}) & \text{on } [t_L,\infty)\times \partial V^{(j)}_{2m}, \\ u^{(2)}(t,\mathbf{y}) &= \mathbb{E}\left[\prod_{\ell=1}^{L}\prod_{i=1}^{n_\ell}\eta_{t_\ell}\left(x_i^{(\ell)}\right)\right] & \text{on } [t_L,\infty)\times\{y_1=y_2\}, \\ u^{(2m)}(t_L,\mathbf{y}) &= & \\ & \mathbb{E}\left[\prod_{\ell=1}^{L}\prod_{i=1}^{n_\ell}\eta_{t_\ell}\left(x_i^{(\ell)}\right)\prod_{j=1}^{m}(-1)^{\eta_{t_L}[y_{2j-1},y_{2j})}\right] & \text{on } V_{2m}. \end{cases}$$

The boundary and initial conditions are immediate. The only part to comment on is the differential equation. Since the process is time-homogeneous and Markov, this follows lemma 4, the single-time ODE characterisation for spin expectations. For example, by the tower property $u^{(2m)}(t,\mathbf{y})$ may be written as an iterated expectation with the inner expectation conditioned on the σ-algebra generated by the process up to time t_L. Pulling the product of intensities outside inner expectation and applying the Markov property at t_L leaves a spin expectation at time $t - t_L$ with initial condition given by the original process at time t_L. Lemma 4 implies both that this initial condition is Pfaffian and then, by its extension to random initials, that the spin expectation solves the differential equation. It remains to exchange the time derivative with the outer expectation, which follows from regularity properties of the spin expectation.

Since the mixed expectation is bounded in absolute value by 1, the reasoning in lemma 2 extends to give unique solvability of the infinite sequence $((\mathrm{ODE})'_{2m} : m = 1, 2, \dots)$, within the class of continuously differentiable functions satisfying

$$\sup_{t\geq t_L}\sup_{\mathbf{y}\in V_{2m}} |u^{(2m)}(t,\mathbf{y})| < \infty.$$

It remains to show that the claimed Pfaffian $(-2)^{-\sum_{\ell=1}^{L} n_\ell}\,\mathrm{Pf}(\tilde{\mathbf{K}}(A))$ is a solution for $t \in [t_L, \infty)$ and $\mathbf{y} \in V_{2m}$, then extend to $\mathbf{y} \in \overline{V}_{2m}$.

Firstly, consider the differential equation in $(\mathrm{ODE})'_{2m}$. By definition the Pfaffian $\mathrm{Pf}(\tilde{\mathbf{K}}(A))$ is a sum of products of the matrix entries, each space-time point of (5.8) appearing exactly once in each product. The entries containing the variables t and $\mathbf{y}$ come from the $\tilde{\mathbf{K}}(t, y_i; t, y_j)$, with $i < j$, and $\tilde{\mathbf{K}}(t_\ell, x; t, y_j)$ terms of (5.9) (and their lower triangular anti-symmetric counterparts). We have already seen in lemma 4 that the spin expectation $\tilde{\mathbf{K}}(t, y_i; t, y_j) = K_t(y_i, y_j)$ solves

$$\partial_t \tilde{\mathbf{K}}(t, y_i; t, y_j) = (L^A_{y_i} + L^A_{y_j})\tilde{\mathbf{K}}(t, y_i; t, y_j).$$

Noting that $D_1^+\big(p_{t-t_\ell} * \tilde{K}_{t_\ell}\big)(x, y_j) = \big(p_{t-t_\ell} * D_1^+ \tilde{K}_{t_\ell}\big)(x, y_j)$, each entry of $\tilde{\mathbf{K}}(t_\ell, x; t, y_j)$ may be expressed as a convolution against p_{t-t_ℓ} and so solves the equation in one spatial dimensional

$$\partial_t \tilde{\mathbf{K}}(t_\ell, x; t, y_j) = L^A_{y_j} \tilde{\mathbf{K}}(t_\ell, x; t, y_j).$$

Hence each product, and therefore the entire Pfaffian, solves the desired equation $\partial_t u = \sum_{j=1}^{2m} L^A_{y_j} u$ for $\mathbf{y} \in V_{2m}$.

Next we check the boundary conditions. For $m > 1$ these follow exactly as for single-time spin expectations. Indeed $\tilde{\mathbf{K}}(t, y_j; t, y_{j+1}) = 1$ if $y_j = y_{j+1}$, and subtracting the y_{j+1}-th row (and column) from the y_j-th, then expanding in the y_j-th row, gives the Pfaffian with these rows and columns removed, which is the desired boundary condition. For $m = 1$ these steps give the Pfaffian

$$\mathrm{Pf}\left(\tilde{\mathbf{K}}\left(t_\ell, x_i^{(\ell)}; t_k, x_j^{(k)}\right) : \ell, k \le L, i \le n_\ell, j \le n_k\right).$$

To complete the boundary conditions, we must show that this Pfaffian is equal to $\mathbb{E}\left[\prod_{\ell=1}^{L} \prod_{i=1}^{n_\ell} \eta_{t_\ell}\left(x_i^{(\ell)}\right)\right]$. Note that this represents exactly the claim of theorem 7, since the $m = 1$ boundary condition corresponds to taking zero spin pairs. The desired equivalence coincides with the $m = 0$ initial condition and we therefore turn attention to establishing the initial conditions for $m \ge 0$.

To prove that the initial conditions are satisfied, we develop the mixed expectation

$$u^{(2m)}(t_L, \mathbf{y}) = \mathbb{E}\left[\prod_{\ell=1}^{L} \prod_{i=1}^{n_\ell} \eta_{t_\ell}\left(x_i^{(\ell)}\right) \prod_{j=1}^{m} (-1)^{\eta_{t_L}[y_{2j-1}, y_{2j})}\right],$$

and show that it coincides with the Pfaffian

$$(-2)^{-\sum_{\ell=1}^{L} n_\ell} \mathrm{Pf}(\tilde{\mathbf{K}}(A_0)) \quad \text{where} \quad A_0 = \underbrace{\bigcup_{\ell=1}^{L} \bigcup_{i=1}^{n_\ell} \left(t_\ell, x_i^{(\ell)}\right)}_{\text{Particles}} \cup \underbrace{\bigcup_{j=1}^{2m} (t_L, y_j)}_{\text{Spins}}. \tag{5.10}$$

The outline is to rewrite the intensities at t_L in terms of spins, in order to express the mixed expectation as a product of intensities at $L - 1$ time slices and spins at a later time. The inductive hypothesis allows us to write such expectations as Pfaffians and we then check that the claimed Pfaffians emerge when the spins at t_L are turned back into intensities. The explicit relation between spins and intensities

is given by (3.14)

$$D_y^+(-1)^{\eta[x,y)}\Big|_{y=x} = (-1)^{\eta[x,x+1)} - 1 = -2\eta(x).$$

Using this we may express the initial condition as

$$u^{(2m)}(t_L, \mathbf{y}) = (-2)^{-n_L} \left(\prod_{i=1}^{n_L} D_{x_i}^+ \mathbb{E}\left[\text{—} \right] \right) \Big|_{x_1 = x_1^{(L)}, \ldots, x_{n_L} = x_{n_L}^{(L)}},$$

where the expectation is given by

$$\mathbb{E}\left[\text{—} \right] = \mathbb{E}\left[\prod_{\ell=1}^{L-1} \prod_{i=1}^{n_\ell} \eta_{t_\ell}\left(x_i^{(\ell)}\right) \prod_{i=1}^{n_L} (-1)^{\eta_{t_L}[x_i^{(L)}, x_i)} \prod_{j=1}^{m} (-1)^{\eta_{t_L}[y_{2j-1}, y_{2j})} \right]. \tag{5.11}$$

By the inductive hypothesis this expectation is given by a Pfaffian of $\tilde{\mathbf{K}}$, but to give the Pfaffian explicitly we must fix an ordering on the set of spin positions $S = \{(t_L, x_i^{(L)}) \cup (t_L, x_i) \cup (t_L, y_j) : 1 \le i \le L, 1 \le j \le 2m\}$ at coincidental points. Note that x_i takes the value $x_i^{(L)}$ or $x_i^{(L)} + 1$. Moreover the underlying t_L particle positions $x_i^{(L)}$ are distinct, so we may assume that the spin positions in the expectation coming from the intensities satisfy

$$x_1^{(L)} \le x_1 \le x_2^{(L)} \le x_2 \le \cdots \le x_{n_L}^{(L)} \le x_{n_L}. \tag{5.12}$$

This gives an ordering on these spin positions, but we must also incorporate the y_j positions. The choice of ordering on coincidental points is arbitrary, but for convenience we fix the ordering $\theta : S \to \{1, 2, \ldots, 2n_L + 2m\}$ in which the (ordered by index) coincidental y_j slot in immediately below the smallest $x_i^{(L)}$ that is larger or equal. An example of this ordering on a set of points is

$$x_1^{(L)} = x_1 < y_1 = y_2 = x_2^{(L)} < x_2 = y_3 = x_3^{(L)} < x_3 < y_4.$$

This choice is practical because the pairs $x_i^{(L)}$ and x_i are kept adjacent. With this choice, the expectation (5.11) is given by the inductive hypothesis as

$$(-2)^{-\sum_{\ell=1}^{L-1} n_\ell} \operatorname{Pf}(\tilde{\mathbf{K}}(A_0')) \quad \text{where} \quad A_0' = \underbrace{\bigcup_{\ell=1}^{L-1} \bigcup_{i=1}^{n_\ell} \left(t_\ell, x_i^{(\ell)}\right)}_{\text{Particles}} \cup \underbrace{S}_{\text{Spins}},$$

with the ordering θ for coincidental spin points. Before recombining the discrete derivatives to restore the initial condition, we manipulate the Pfaffians by exchanging rows and columns to obtain the desired ordering, in which the (ordered) $x_i^{(L)}$ and x_i come before the (ordered) y_j. Exchanging a pair of rows (and corresponding columns) corresponds to conjugation by an elementary matrix with determinant -1, and so flips the sign of the Pfaffian. The ordering θ was chosen as it takes an even number of transpositions to reach the desired ordering. Indeed, since the pairs $x_i^{(L)}$ and x_i are adjacent, it takes an even number of transpositions to move y_{2m} to the end, then an even number to move y_{2m-1} immediately before y_{2m}, then an even number to move y_{2m-2} immediately before y_{2m-1}, and so on. In terms of particle positions, acting spins arising from particle positions and genuine spins, the resulting Pfaffian expression for the expectation (5.11) has the block form

$$(-2)^{-\sum_{\ell=1}^{L-1} n_\ell} \operatorname{Pf} \left(\begin{array}{c|c|c} M_{\text{p-p}} & M_{\text{p-a}} & M_{\text{p-s}} \\ \hline -M_{\text{p-a}}^T & M_{\text{a-a}} & M_{\text{a-s}} \\ \hline -M_{\text{p-s}}^T & -M_{\text{a-s}}^T & M_{\text{s-s}} \end{array} \right), \tag{5.13}$$

where each block has the following entries from (5.9)

- $M_{\text{p-p}}$ is the $2\sum_{\ell=1}^{L-1} n_\ell \times 2\sum_{\ell=1}^{L-1} n_\ell$ anti-symmetric matrix containing the 2×2 (particle, particle) entries $\tilde{\mathbf{K}}(t_\ell, x_i^{(\ell)}; t_k, x_j^{(k)})$;
- $M_{\text{p-a}}$ is the $2\sum_{\ell=1}^{L-1} n_\ell \times 2n_L$ matrix containing the 2×1 (particle, acting spin) entries $\tilde{\mathbf{K}}(t_\ell, x_i^{(\ell)}; t_L, x_j^{(L)})$ and $\tilde{\mathbf{K}}(t_\ell, x_i^{(\ell)}; t_L, x_j)$;
- $M_{\text{p-s}}$ is the $2\sum_{\ell=1}^{L-1} n_\ell \times 2m$ matrix containing the 2×1 (particle, spin) entries $\tilde{\mathbf{K}}(t_\ell, x_i^{(\ell)}; t_L, y_j)$;
- $M_{\text{a-a}}$ is the $2n_L \times 2n_L$ anti-symmetric matrix containing the 1×1 (acting spin, acting spin) entries $\tilde{\mathbf{K}}(t_L, x_i^{(L)}; t_L, x_j^{(L)})$, $\tilde{\mathbf{K}}(t_L, x_i^{(L)}; t_L, x_j)$, $\tilde{\mathbf{K}}(t_L, x_i; t_L, x_j^{(L)})$ and $\tilde{\mathbf{K}}(t_L, x_i; t_L, x_j)$;
- $M_{\text{a-s}}$ is the $2n_L \times 2m$ matrix containing the 1×1 (acting spin, spin) entries $\tilde{\mathbf{K}}(t_L, x_i^{(L)}; t_L, y_j)$ and $\tilde{\mathbf{K}}(t_L, x_i; t_L, y_j)$;
- $M_{\text{s-s}}$ is the $2m \times 2m$ anti-symmetric matrix containing the 1×1 (spin, spin) entries $\tilde{\mathbf{K}}(t_L, y_i; t_L, y_j)$.

The entries of $M_{\text{a-a}}$, $M_{\text{a-s}}$ and $M_{\text{s-s}}$ are given by

$$\tilde{\mathbf{K}}(t_L, w_1; t_L, w_2) = K_{t_L}(\min\{w_1, w_2\}, \max\{w_1, w_2\}) \operatorname{sgn}(\theta(w_2) - \theta(w_1)),$$

for (t_L, w_1), $(t_L, w_2) \in S$. Note that θ respects the ordering on $\bigcup_{j=1}^{2m}(t_L, y_j)$, so the matrix $M_{\text{s-s}}$ is the spin matrix with entries

$$\tilde{\mathbf{K}}(t_L, y_i; t_L, y_j) = K_t(\min\{y_i, y_j\}, \max\{y_i, y_j\}) \operatorname{sgn}(j - i).$$

Moreover the ordering (5.12) is respected by θ and the matrix $M_{\text{a-a}}$ is also a spin matrix, with upper triangular entries

$$\tilde{\mathbf{K}}(t_L, w_1; t_L, w_2) = K_{t_L}(w_1, w_2),$$

for (t_L, w_1), $(t_L, w_2) \in \bigcup_{i=1}^{n_L}(t_L, x_i^{(L)}) \cup (t_L, x_i)$. Unpicking the ordering, the entries of $M_{\text{a-s}}$ are given in terms of the anti-symmetrised function $\tilde{K}_{t_L}$ by

$$\begin{aligned}
\tilde{\mathbf{K}}(t_L, x_i^{(L)}; t_L, y_j) &= \tilde{K}_{t_L}(x_i^{(L)}, y_j) - 1(x_i^{(L)} = y_j), \\
\tilde{\mathbf{K}}(t_L, x_i; t_L, y_j) &= \tilde{K}_{t_L}(x_i, y_j) + 1(x_i = y_j)\left(1(x_i = x_i^{(L)} + 1) - 1(x_i = x_i^{(L)})\right).
\end{aligned} \tag{5.14}$$

It remains to apply the discrete derivatives and evaluations to (5.13) to restore the initial condition, then show that this coincides with the claimed Pfaffian expression (5.10). As explained in the proof of theorem 1, each discrete derivative may be passed onto the row and column on which it acts. All together, applying the derivatives and evaluations, the initial condition is given by

$$u^{(2m)}(t_L, \mathbf{y}) = (-2)^{-\sum_{\ell=1}^{L} n_\ell} \operatorname{Pf} \left(\begin{array}{c|c|c} M_{\text{p-p}} & \tilde{M}_{\text{p-a}} & M_{\text{p-s}} \\ \hline -\tilde{M}_{\text{p-a}}^T & \tilde{M}_{\text{a-a}} & \tilde{M}_{\text{a-s}} \\ \hline -M_{\text{p-s}}^T & -\tilde{M}_{\text{a-s}}^T & M_{\text{s-s}} \end{array} \right),$$

where

- $\tilde{M}_{\text{p-a}}$ is obtained from $M_{\text{p-a}}$ by replacing

  $$\tilde{\mathbf{K}}(t_\ell, x_i^{(\ell)}; t_L, x_j) \mapsto D_2^+ \tilde{\mathbf{K}}(t_\ell, x_i^{(\ell)}; t_L, x_j^{(L)}),$$

  where the operator acts on each entry of the (2×1) term;

- $\tilde{M}_{\text{a-a}}$ is obtained from $M_{\text{a-a}}$ by replacing

$$\begin{aligned}\tilde{\mathbf{K}}(t_L, x_i^{(L)}; t_L, x_j) &\mapsto D_2^+\tilde{\mathbf{K}}(t_L, x_i^{(L)}; t_L, x_j^{(L)}),\\ \tilde{\mathbf{K}}(t_L, x_i; t_L, x_j^{(L)}) &\mapsto D_1^+\tilde{\mathbf{K}}(t_L, x_i^{(L)}; t_L, x_j^{(L)}),\\ \tilde{\mathbf{K}}(t_L, x_i; t_L, x_j) &\mapsto D_1^+D_2^+\tilde{\mathbf{K}}(t_L, x_i^{(L)}; t_L, x_j^{(L)});\end{aligned}$$

- $\tilde{M}_{\text{a-s}}$ is obtained from $M_{\text{a-s}}$ by replacing

$$\tilde{\mathbf{K}}(t_L, x_i; t_L, y_j) \mapsto D_1^+\tilde{\mathbf{K}}(t_L, x_i^{(L)}; t_L, y_j).$$

The matrices $M_{\text{p-p}}$ and $M_{\text{p-s}}$, which do not involve acting spins, are trivially in the desired form, and we have seen that so is $M_{\text{s-s}}$. It remains to deal with the acting spin entries of $\tilde{M}_{\text{p-a}}$, $\tilde{M}_{\text{a-a}}$ and $\tilde{M}_{\text{a-s}}$. In particular, referring to (5.9), we must show that the acting spin entries combine to form the corresponding particle entries in (5.10). Firstly, for $\tilde{M}_{\text{p-a}}$ the entries $\tilde{\mathbf{K}}(t_\ell, x_i^{(\ell)}; t_L, x_j^{(L)})$ and $D_2^+\tilde{\mathbf{K}}(t_\ell, x_i^{(\ell)}; t_L, x_j^{(L)})$ are given by

$$\tilde{\mathbf{K}}(t_\ell, w; t_L, z) = \begin{pmatrix} \Big(p_{t_L-t_\ell} * \tilde{K}_{t_\ell}\Big)(w,z) - p_{t_L-t_\ell}(w,z) \\ D_1^+\Big(p_{t_L-t_\ell} * \tilde{K}_{t_\ell}\Big)(w,z) + p_{t_L-t_\ell}(w,z-1) + p_{t_L-t_\ell}(w,z)\end{pmatrix},$$

$$D_2^+\tilde{\mathbf{K}}(t_\ell, w; t_L, z) = \begin{pmatrix} D_2^+\Big(p_{t_L-t_\ell} * \tilde{K}_{t_\ell}\Big)(w,z) - p_{t_L-t_\ell}(w,z+1) + p_{t_L-t_\ell}(w,z) \\ D_2^+D_1^+\Big(p_{t_L-t_\ell} * \tilde{K}_{t_\ell}\Big)(w,z) + p_{t_L-t_\ell}(w,z+1) - p_{t_L-t_\ell}(w,z-1)\end{pmatrix},$$

where $w = x_i^{(\ell)}$ and $z = x_j^{(L)}$. The 2×2 block formed by combining these terms is the desired (particle, particle) entry $\tilde{\mathbf{K}}(t_\ell, x_i^{(\ell)}; t_L, x_j^{(L)})$ of (5.10) (taking into account the factor of -2 for $\tilde{\mathbf{K}}$). Next the entries of $\tilde{M}_{\text{a-a}}$ combine to give, for $x_i^{(L)} < x_j^{(L)}$, the upper triangular block

$$\begin{pmatrix} \tilde{\mathbf{K}}(t_L, x_i^{(L)}; t_L, x_j^{(L)}) & D_2^+\tilde{\mathbf{K}}(t_L, x_i^{(L)}; t_L, x_j^{(L)}) \\ D_1^+\tilde{\mathbf{K}}(t_L, x_i^{(L)}; t_L, x_j^{(L)}) & D_1^+D_2^+\tilde{\mathbf{K}}(t_L, x_i^{(L)}; t_L, x_j^{(L)})\end{pmatrix} = \begin{pmatrix} K_{t_L}(x_i^{(L)}, x_j^{(L)}) & D_2^+K_{t_L}(x_i^{(L)}, x_j^{(L)}) \\ D_1^+K_{t_L}(x_i^{(L)}, x_j^{(L)}) & D_1^+D_2^+K_{t_L}(x_i^{(L)}, x_j^{(L)})\end{pmatrix},$$

and $D_2^+\tilde{\mathbf{K}}(t_L, x_i^{(L)}; t_L, x_i^{(L)}) = D_2^+K_{t_L}(x_i^{(L)}, x_i^{(L)})$. These blocks coincide with the corresponding single-time (particle, particle) entries $\tilde{\mathbf{K}}(t_L, x_i^{(L)}; t_L, x_j^{(L)})$ of (5.10), and the lower triangular blocks are determined by anti-symmetry. Finally, appealing

to (5.14), the entries of $\tilde{M}_{\text{a-s}}$ combine to give

$$\begin{pmatrix} \tilde{\mathbf{K}}(t_L, x_i^{(L)}; t_L, y_j) \\ D_1^+ \tilde{\mathbf{K}}(t_L, x_i^{(L)}; t_L, y_j) \end{pmatrix} = \begin{pmatrix} \tilde{K}_{t_L}(x_i^{(L)}, y_j) - 1(x_i^{(L)} = y_j) \\ D_1^+ \tilde{K}_{t_L}(x_i^{(L)}, y_j) + 1(x_i^{(L)} + 1 = y_j) + 1(x_i^{(L)} = y_j) \end{pmatrix},$$

the desired 2×1 (particle, spin) entry $\tilde{\mathbf{K}}(t_L, x_i^{(L)}; t_L, y_j)$ of (5.10). This completes the proof that the mixed expectation initial condition $u^{(2m)}(t_L, \mathbf{y})$ coincides with the Pfaffian initial condition (5.10). Moreover, by the above remarks, taking $m = 0$ finishes the proof of the boundary conditions.

Finally, the entries of $\tilde{\mathbf{K}}(A)$ in (5.7) are uniformly bounded in absolute value. Indeed each entry may be written as a finite sum of terms $(p_r * f)(w, z)$ for some $r \geq 0$, $f : \mathbb{Z}^2 \to \mathbb{R}$ with $\|f\|_{l^\infty} \leq 1$ and $w, z \in \mathbb{Z}$. For example, expanding the discrete derivatives and writing $p_r(x, y) = \sum_{z \in \mathbb{Z}} p_r(z, y) 1(x = z) = (p_r * 1(\cdot))(x, y)$, the $(2, 2)$ (particle-particle) entry of (5.9) is given by

$$\begin{aligned} -2\mathbf{K}_{22}(t_\ell, x; t_L, y) = &\left(p_{t_L - t_\ell} * \tilde{K}_{t_\ell}\right)(x+1, y+1) + \left(p_{t_L - t_\ell} * \tilde{K}_{t_\ell}\right)(x, y) \\ &- \left(p_{t_L - t_\ell} * \tilde{K}_{t_\ell}\right)(x+1, y) - \left(p_{t_L - t_\ell} * \tilde{K}_{t_\ell}\right)(x, y+1) \\ &+ \left(p_{t_L - t_\ell} * 1(\cdot)\right)(x, y+1) - \left(p_{t_L - t_\ell} * 1(\cdot)\right)(x, y-1). \end{aligned}$$

Noting that $|\tilde{K}_r(x, y)| \leq 1$ the triangle inequality gives $| - 2\mathbf{K}_{22}(t_\ell, x; t_L, y)| \leq 6(p_r * 1)$. The convolution $(p_r * 1)$ may be interpreted as a probability for the random walk with generator L^A, so is bounded by 1. (More precisely, $(p_r * 1)$ equals the probability that the random walk with generator $q_x D^+ + p_x D^-$, killed at rate $2m_x$, survives till time r.) It follows that the Pfaffian $(-2)^{-\sum_{\ell=1}^L n_\ell} \operatorname{Pf}(\tilde{\mathbf{K}}(A))$ is a uniformly bounded function on $[t_L, \infty) \times V_{2m}$. The aforementioned uniqueness of solutions gives the desired identity (5.7) for $\mathbf{y} \in V_{2m}$. To complete the induction it remains to show that equality also holds on $\overline{V}_{2m}$. This extension may be proved exactly as in lemma 2, by repeating the argument for the boundary conditions to iteratively remove any equalities in $y_1 \leq y_2 \leq \cdots \leq y_{2m}$ until

$$u^{(2m)}(t, \mathbf{y}) = u^{(2k)}(t, \mathbf{z}) \qquad \text{and} \qquad \operatorname{Pf}(\tilde{\mathbf{K}}(A)) = \operatorname{Pf}(\tilde{\mathbf{K}}(A_{\mathbf{z}})),$$

for some subset $\mathbf{z} = (z_1, \ldots, z_k) \in V_{2k}$ of $\mathbf{y}$ with $k \leq m$ and

$$A_{\mathbf{z}} = \underbrace{\bigcup_{\ell=1}^{L} \bigcup_{i=1}^{n_\ell} \left(t_\ell, x_i^{(\ell)}\right)}_{\text{Particles}} \cup \underbrace{\bigcup_{j=1}^{k} (t_L, z_j)}_{\text{Spins}}.$$

Equality on the larger set follows and the proof of theorem 8 is complete. □

Example 13. Consider example 8 of annihilating symmetric random walks with homogeneous pairwise immigration, defined by setting $q_x = p_x = 1$ and $m_x = m$ for some $m > 0$, and initial condition $\eta_0(x)$ product Bernoulli(λ) for some $\lambda \in (0, 1]$. By theorem 7 and remark 23 the system is an extended Pfaffian point process with kernel $\mathbf{K}(s, x; t, y)$ defined by (5.5) and (5.6), where $K_t(x, y)$ is given by (3.26) and the Green's function $p_t : \mathbb{Z}^2 \to \mathbb{R}$ is defined in terms of the modified Bessel function of the first kind by

$$p_t(x, y) = e^{-2(1+m)t} I_{y-x}(2t), \qquad \text{where} \quad I_x(t) = \frac{1}{\pi} \int_0^\pi e^{t\cos(w)} \cos(xw)\, \mathrm{d}w.$$

Remark 27. By considering asymptotics of an ARW system, we compare the extended kernel $\mathbf{K}$ in theorem 7 with that given in [58] for annihilating Brownian motions under a maximal entrance law. The underlying ARW process $(X_t : t \geq 0)$ on $\mathbb{Z}$ is defined by setting $p_x = q_x = 1/2$, $m_x = 0$ and $\eta_0(x)$ to be independent Bernoulli(λ) for some $\lambda \in (0, 1]$. The diffusive scaling theory for the single-time projection X_t is developed in section 4.2.1 (taking $\theta = 1$ and scaling time $t \mapsto \frac{t}{2}$). This scaling regime is suitable for the whole temporal process and for $\epsilon > 0$ we define $(X_t^{(\epsilon)} : t \geq 0)$ on $\epsilon\mathbb{Z}$ by

$$X_t^{(\epsilon)}(\mathrm{d}x) = X_{\epsilon^{-2}t}(\epsilon^{-1}\mathrm{d}x) \qquad \text{on } \epsilon\mathbb{Z}.$$

The process $(X_t^{(\epsilon)} : t \geq 0)$ is an extended Pfaffian point process on $\epsilon\mathbb{Z}$ with kernel $\mathbf{K}^{(\epsilon)}(s, x; t, y)$. Appealing to (5.6) and the basic scaled kernel (4.9), the entries of the extended kernel $\mathbf{K}^{(\epsilon)}(s, x; t, y)$ for $s < t$ and $x, y \in \epsilon\mathbb{Z}$ are given explicitly in terms of the single-time kernel function $K_t^{(\epsilon)}(x, y)$ in (4.11) and the scaled Green's

function $p_t^{(\epsilon)}(x,y) = p_{\epsilon^{-2}t}(\epsilon^{-1}x, \epsilon^{-1}y)$ by

$$\begin{aligned}
\mathbf{K}_{11}^{(\epsilon)}(s,x;t,y) &= \textstyle\sum_{z\in\epsilon\mathbb{Z}} p_{t-s}^{(\epsilon)}(z,y)\left(-\frac{1}{2}K_s^{(\epsilon)}(x,z)\right) + \frac{1}{2}p_{t-s}^{(\epsilon)}(x,y),\\
\mathbf{K}_{12}^{(\epsilon)}(s,x;t,y) &= \textstyle\sum_{z\in\epsilon\mathbb{Z}} p_{t-s}^{(\epsilon)}(z,y)\left(-\frac{\epsilon}{2}D_\epsilon^{(0,1)}K_s^{(\epsilon)}(x,z)\right) - p_{t-s}^{(\epsilon)}(x,y),\\
\mathbf{K}_{21}^{(\epsilon)}(s,x;t,y) &= \textstyle\sum_{z\in\epsilon\mathbb{Z}} p_{t-s}^{(\epsilon)}(z,y)\left(-\frac{\epsilon}{2}D_\epsilon^{(1,0)}K_s^{(\epsilon)}(x,z)\right),\\
\mathbf{K}_{22}^{(\epsilon)}(s,x;t,y) &= \textstyle\sum_{z\in\epsilon\mathbb{Z}} p_{t-s}^{(\epsilon)}(z,y)\left(-\frac{\epsilon^2}{2}D_\epsilon^{(1,1)}K_s^{(\epsilon)}(x,z)\right).
\end{aligned}$$

Redistributing factors of ϵ gives an alternative extended kernel $\tilde{\mathbf{K}}^{(\epsilon)}(s,x;t,y)$, with entries given in terms of the single-time kernel $\mathbf{K}_t^{(\epsilon)}(x,y)$ in (4.10) by

$$\begin{aligned}
\tilde{\mathbf{K}}_{11}^{(\epsilon)}(s,x;t,y) &= \epsilon\textstyle\sum_{z\in\epsilon\mathbb{Z}} p_{t-s}^{(\epsilon)}(z,y)(\mathbf{K}_s^{(\epsilon)})_{11}(x,z) + \frac{\epsilon}{2}p_{t-s}^{(\epsilon)}(x,y),\\
\tilde{\mathbf{K}}_{12}^{(\epsilon)}(s,x;t,y) &= \epsilon\textstyle\sum_{z\in\epsilon\mathbb{Z}} p_{t-s}^{(\epsilon)}(z,y)(\mathbf{K}_s^{(\epsilon)})_{12}(x,z) - p_{t-s}^{(\epsilon)}(x,y),\\
\tilde{\mathbf{K}}_{21}^{(\epsilon)}(s,x;t,y) &= \epsilon\textstyle\sum_{z\in\epsilon\mathbb{Z}} p_{t-s}^{(\epsilon)}(z,y)(\mathbf{K}_s^{(\epsilon)})_{21}(x,z),\\
\tilde{\mathbf{K}}_{22}^{(\epsilon)}(s,x;t,y) &= \epsilon\textstyle\sum_{z\in\epsilon\mathbb{Z}} p_{t-s}^{(\epsilon)}(z,y)(\mathbf{K}_s^{(\epsilon)})_{22}(x,z).
\end{aligned}$$

With the single-time theory in mind, the candidate limit kernel is given by taking limits of $\epsilon^{-1}\tilde{\mathbf{K}}^{(\epsilon)}$, scaled to balance the mean number of particles per unit interval at each time, which has entries

$$\begin{aligned}
\epsilon^{-1}\tilde{\mathbf{K}}_{11}^{(\epsilon)}(s,x;t,y) &= \epsilon\textstyle\sum_{z\in\epsilon\mathbb{Z}}\left(\epsilon^{-1}p_{t-s}^{(\epsilon)}(z,y)\right)\left(\epsilon^{-1}(\mathbf{K}_s^{(\epsilon)})_{11}(x,z)\right) + \frac{1}{2}p_{t-s}^{(\epsilon)}(x,y),\\
\epsilon^{-1}\tilde{\mathbf{K}}_{12}^{(\epsilon)}(s,x;t,y) &= \epsilon\textstyle\sum_{z\in\epsilon\mathbb{Z}}\left(\epsilon^{-1}p_{t-s}^{(\epsilon)}(z,y)\right)\left(\epsilon^{-1}(\mathbf{K}_s^{(\epsilon)})_{12}(x,z)\right) - \epsilon^{-1}p_{t-s}^{(\epsilon)}(x,y),\\
\epsilon^{-1}\tilde{\mathbf{K}}_{21}^{(\epsilon)}(s,x;t,y) &= \epsilon\textstyle\sum_{z\in\epsilon\mathbb{Z}}\left(\epsilon^{-1}p_{t-s}^{(\epsilon)}(z,y)\right)\left(\epsilon^{-1}(\mathbf{K}_s^{(\epsilon)})_{21}(x,z)\right),\\
\epsilon^{-1}\tilde{\mathbf{K}}_{22}^{(\epsilon)}(s,x;t,y) &= \epsilon\textstyle\sum_{z\in\epsilon\mathbb{Z}}\left(\epsilon^{-1}p_{t-s}^{(\epsilon)}(z,y)\right)\left(\epsilon^{-1}(\mathbf{K}_s^{(\epsilon)})_{22}(x,z)\right).
\end{aligned}$$

Convergence of the one-dimensional marginals is proved in section 4.2. Theorem 5 gives that $X_t^{(\epsilon)}$ converges in distribution to X_t^c with kernel $\mathbf{K}_t^c(x,y)$ given by

$$\mathbf{K}_t^c(x,y) = -\begin{pmatrix} F_t(|y-x|)\operatorname{sgn}(y-x) & F_t'(y-x) \\ -F_t'(y-x) & -F_t''(y-x)\end{pmatrix},$$

for x, $y \in \mathbb{R}$ where $F_t(z) = 2^{-1} \operatorname{erfc}\left(z/2\sqrt{t}\right)$. In particular, the proof gives convergence of the kernels $\epsilon^{-1}(\mathbf{K}_s^{(\epsilon)})_{ij} \to (\mathbf{K}_s^c)_{ij}$. A local central limit theorem gives convergence of the scaled Green's functions to the continuum heat kernel $\epsilon^{-1} p_t^{(\epsilon)}(x_\epsilon, y_\epsilon) \to p_t^c(y-x) = (2\pi t)^{-1/2} e^{-(y-x)^2/2t}$ for $x_\epsilon \to x$, $y_\epsilon \to y$. This implies moreover that the additional $(1,1)$ term of the extended kernel $\epsilon^{-1}\tilde{\mathbf{K}}^{(\epsilon)}(s,x;t,y)$, namely $\frac{1}{2}p_t^{(\epsilon)}(x_\epsilon, y_\epsilon)$, vanishes in the limit. Finally, the summations in $\epsilon^{-1}\tilde{\mathbf{K}}^{(\epsilon)}(s,x;t,y)$ are Riemann approximations to integrals and all together the candidate limit extended kernel $\mathbf{K}^c(s,x;t,y)$ is given, for $s < t$ and x, $y \in \mathbb{R}$, by

$$\mathbf{K}_{ij}^c(s,x;t,y) = \left(p_{t-s}^c \star (\mathbf{K}_s^c)_{ij}\right)(x,y) - 1(i=1, j=2) p_{t-s}^c(y-x),$$

where the one-dimensional convolution makes because $\mathbf{K}_s^c(x,y)$ depends on $y - x$. The extended kernel $\mathbf{K}^c$ coincides with $\tilde{\mathbf{K}}^{\mathrm{ABM}}$, given by (5.2), for annihilating Brownian motions under a maximal entrance law (after swapping the order of entries by proposition 12 part 4). See example 10 for more details. Note that we have not proved convergence of the scaled processes $(X_t^{(\epsilon)} : t \geq 0)$, we have only shown convergence of the extended kernel entries. It is natural to conjecture that the scaled ARW system converges to annihilating Brownian motions in a stronger process sense. In fact, we could repeat the above analysis for scaled independent Bernoulli initial conditions appearing in section 4.2.2. The limit extended kernel has the same form but is defined by the single-time kernel in theorem 6. Building on the single-time section, the corresponding conjecture is that the limit process is annihilating Brownian motions started from rate μ Poisson initial conditions.

Remark 28. The analysis in remark 27 may be extended to ARWI processes with constant immigration, such as example 13. We scale the discrete processes as for single times in section 4.3, namely with $p_x^{(\epsilon)} = q_x^{(\epsilon)} = 1/2$, $m^{(\epsilon)} = \frac{\epsilon^2 c}{2}$ for some $c \geq 0$ and $\lambda \in (0,1]$ for the initial condition Bernoulli rate. The candidate limit extended kernel $\mathbf{K}^c(s,x;t,y)$ is given for $s = t$ by conjecture 1, namely for $x < y$

$$\mathbf{K}_{ij}^c(s,x;t,y) = \mathbf{K}_t^c(x,y) = -\frac{1}{2}\begin{pmatrix} K_t^c(x,y) & \partial_2 K_t^c(x,y) \\ \partial_1 K_t^c(x,y) & \partial_1\partial_2 K_t^c(x,y) \end{pmatrix},$$

and $\mathbf{K}_{12}^c(s,x;t,x) = (\mathbf{K}_t^c)_{12}(x,x) = -\frac{1}{2}\partial_2 K_t^c(x,x)$, where $K_t^c(x,y)$ is defined by (4.44) and only depends on $y - x$. The extended kernel for $s < t$ is then

$$\mathbf{K}_{ij}^c(s,x;t,y) = \left(e^{-c(t-s)} p_{t-s}^c \star (\mathbf{K}_s^c)_{ij}\right)(x,y) - 1(i=1, j=2) e^{-c(t-s)} p_{t-s}^c(y-x),$$

where $p_t^c(z) = (2\pi t)^{-1/2} e^{-z^2/2t}$ is the heat kernel and the one-dimensional convo-

lution makes sense because $\mathbf{K}^c_s(x,y)$ depends on $y-x$. We expect that the scaled ARWI processes converge to a stochastic process whose finite-dimensional distributions are characterised by the extended kernel $\mathbf{K}^c(s,x;t,y)$. There is not a natural Brownian motion system to conjecture as the limit. Due to infinitesimal immigration we do not even expect the paths to be càdlàg.

The Brownian web [5, 22, 47, 56] is a system of instantly coalescing Brownian motions started from all points in time and space. By restricting to certain paths, the Brownian web contains a wealth of interacting particle systems. For example, annihilating systems may be obtained from coalescing systems by thinning. In fact, there are points of the Brownian web where two trajectories begin at the same point, so-called $(0,2)$-points, which are dense in $\mathbb{R}\times\{t\}$ for each time t (see [22, 47]). These could represent infinitesimal pairwise immigration and we see how the ARWI limit may also be contained the Brownian web.

Given convergence of the finite-dimensional distributions, to construct a limit process it suffices to prove tightness of the approximating processes in a suitable space. Theory for the space of processes with càdlàg paths is well developed and there are simple tightness criteria, however we expect non-càdlàg paths for our process. Suppose that $(X^c_t : t \geq 0)$ is a stochastic process with finite-dimensional distributions characterised by the extended kernel $\mathbf{K}^c(s,x;t,y)$. Consider the product measure $Y_t = X_t \times X_t$ on $\mathbb{R}^2$ and let ϕ be a continuous test function with compact support. If ϕ is supported away from the diagonal then $Y_t(\phi)$ is well behaved when an infinitesimal pair is immigrated. The continuity of $t \mapsto Y_t(\phi)$ may be investigated as the support of ϕ is extended to the diagonal, leading to the theory of weighted topologies. As a first step, there is a tightness criterion of Kurtz [36] in the space of simple processes with measurable paths. This translates into a uniform convergence condition on the extended kernels of the approximating discrete processes.

In [24] an alternative approach to the continuum process is considered, as a limit of annihilating Brownian motion processes with pairwise immigration. The approximating models have immigration of pairs with initial separation ϵ and pair centres distributed according to rate θ_ϵ Poisson process. To obtain a non-degenerate limit θ_ϵ is scaled so that $\epsilon\theta_\epsilon \to \theta$ as $\epsilon \downarrow 0$. This gives a continuum limit process with parameter θ which has equivalent Pfaffian one-dimensional marginals under a maximal entrance law to the ARWI limit in conjecture 1. The approximating continuum processes however are not Pfaffian at each fixed time.

Appendix A

Uniqueness for ODEs

We show that the infinite systems of ODEs appearing in chapter 3 are uniquely solvable within the class of functions with exponential growth. This follows from standard (weighted) Gronwall estimates. A function $f : \mathbb{R}^d \to \mathbb{R}$ satisfying $|f(\mathbf{x})| \leq C_1 e^{C_0|\mathbf{x}|}$ for some constants C_0, $C_1 > 0$, where $|(x_1, \ldots, x_d)| = \sum_{i=1}^d |x_i|$, is said have *exponential growth (of rate C_0)*. To write down the ODE system, we introduce a one-particle generator L given, for $f : \mathbb{Z} \to \mathbb{R}$, by

$$Lf(x) = a(x)\Delta^{(1)}f(x) + b(x)\nabla^{(1)}f(x) - c(x)f(x),$$

where the central discrete operators are defined by (4.34). The coefficients satisfy $a(x) > 0$, $b(x) \in \mathbb{R}$, $c(x) \geq 0$ and the uniform bound $|a(x)| + |b(x)| \leq M$ for some $M > 0$. Fixing $C_0^{(n)} > 0$, we consider the sequence $((\mathrm{ODE})_{2n} : n = 1, 2, \ldots)$ defined, in the framework of lemma 2, by

$$(\mathrm{ODE})_{2n} \qquad \begin{cases} \partial_t u^{(2n)}(t, \mathbf{x}) &= \sum_{i=1}^{2n} L_{x_i} u^{(2n)}(t, \mathbf{x}) & \text{on } [0, \infty) \times V_{2n}, \\ u^{(2n)}(t, \mathbf{x}) &= u^{(2n-2)}(t, \mathbf{x}^{i,i+1}) & \text{on } [0, \infty) \times \partial V_{2n}^{(i)}, \\ u^{(2n)}(0, \mathbf{x}) &= h^{(2n)}(\mathbf{x}) & \text{on } V_{2n}, \end{cases}$$

where $u^{(0)} = 1$ and for each n the function $h^{(2n)}$ has exponential growth of rate $C_0^{(n)}$. We prove the following uniqueness result.

Lemma 11. *The sequence $((ODE)_{2n} : n = 1, 2, \ldots)$ is uniquely solvable within the class of functions with exponential growth. For each n the rate of exponential growth may be taken as $C_0^{(n)}$.*

Note that the ODE systems for CARW, ARWI and BCRW in chapter 3 all fall into this framework.

Proof of lemma 11. We prove the result inductively, since once $(\mathrm{ODE})_{2n}$ is shown to be uniquely solvable, the next order system $(\mathrm{ODE})_{2n+2}$ has well-defined boundary functions and is well-posed. To wit, fix $n \geq 1$ and assume that $(\mathrm{ODE})_{2m}$ is uniquely solvable within the class of functions with exponential growth of rate $C_0^{(m)}$ for $m < n$. The system $(\mathrm{ODE})_{2n}$ is well-posed. Suppose that $(\mathrm{ODE})_{2n}$ has two solutions $u^{(2n)}(t, \mathbf{x})$ and $v^{(2n)}(t, \mathbf{x})$ with exponential growth of rate $C_0^{(n)}$. Denote the difference by $w(t, \mathbf{x}) = u^{(2n)}(t, \mathbf{x}) - v^{(2n)}(t, \mathbf{x})$. It suffices to show that $w^{(2n)}(t, \mathbf{x}) = 0$ for all $t \in [0, \infty)$ and $\mathbf{x} \in V_{2n}$. Note that $w^{(2n)}(t, \mathbf{x})$ solves the equation

$$\left\{ \begin{array}{rcll} \partial_t w^{(2n)}(t, \mathbf{x}) & = & \sum_{i=1}^{2n} L_{x_i} w^{(2n)}(t, \mathbf{x}) & \text{on } [0, \infty) \times V_{2n}, \\ w^{(2n)}(t, \mathbf{x}) & = & 0 & \text{on } [0, \infty) \times \partial V_{2n}^{(i)}, \\ w^{(2n)}(0, \mathbf{x}) & = & 0 & \text{on } V_{2n}, \end{array} \right.$$

Consider the energy functional $E(t) : [0, \infty) \to [0, \infty)$ defined by

$$E(t) = \sum_{\mathbf{x} \in V_{2n}} (w^{(2n)}(t, \mathbf{x}))^2 e^{-\gamma^{(n)}|\mathbf{x}|},$$

for some $\gamma^{(n)} > 2C_0^{(n)}$. The last condition ensures that $E(t)$ is finite, since the weight $\gamma^{(n)}$ sufficiently compensates exponential growth of rate $C_0^{(n)}$. It suffices to show that $E(t) = 0$ for all $t \in [0, \infty)$. This is achieved by bounding $E(t)$ in terms of itself and using Gronwall's inequality. To this end, differentiating with respect to t and developing using the differential equation for $w^{(2n)}(t, \mathbf{x})$

$$\begin{aligned} \frac{\partial}{\partial t} E(t) &= \sum_{\mathbf{x} \in V_{2n}} 2 w^{(2n)}(t, \mathbf{x}) e^{-\gamma^{(n)}|\mathbf{x}|} \frac{\partial}{\partial t} w^{(2n)}(t, \mathbf{x}) \\ &= \sum_{\mathbf{x} \in V_{2n}} 2 w^{(2n)}(t, \mathbf{x}) e^{-\gamma^{(n)}|\mathbf{x}|} \sum_{i=1}^{2n} L_{x_i} w^{(2n)}(t, \mathbf{x}) \\ &= \sum_{i=1}^{2n} \sum_{x_j : j \neq i} e^{-\gamma^{(n)} \sum_{j \neq i} |x_j|} \sum_{x_i} 2 w^{(2n)}(t, \mathbf{x}) e^{-\gamma^{(n)}|x_i|} L_{x_i} w^{(2n)}(t, \mathbf{x}), \end{aligned}$$

where $\sum_{x_j : j \neq i}$ indicates the sum over the variables excluding x_i and $\sum_{x_i} = \sum_{x_i = x_{i-1}+1}^{x_{i+1}-1}$, with the understanding that in the cases $i = 1$ and $2n$ the limits may be $\pm\infty$.

Substituting in for L, the inner summation is given by three terms

$$\begin{aligned}\sum_{x_i} 2w^{(2n)}(t,\mathbf{x})e^{-\gamma^{(n)}|x_i|}L_{x_i}w^{(2n)}(t,\mathbf{x}) &= \sum_{x_i} 2w^{(2n)}(t,\mathbf{x})e^{-\gamma^{(n)}|x_i|}a(x_i)\Delta^{(1)}_{x_i}w^{(2n)}(t,\mathbf{x}) \\ &+ \sum_{x_i} 2w^{(2n)}(t,\mathbf{x})e^{-\gamma^{(n)}|x_i|}b(x_i)\nabla^{(1)}_{x_i}w^{(2n)}(t,\mathbf{x}) \\ &- \sum_{x_i} 2w^{(2n)}(t,\mathbf{x})e^{-\gamma^{(n)}|x_i|}c(x_i)w^{(2n)}(t,\mathbf{x}).\end{aligned}$$

The $c(x_i)$ sum is non-positive and may be discarded in pursuit of an upper bound. Recall the notation $\mathbf{x}^{i\pm}$ for the vector $\mathbf{x}$ with the i-th variable incremented by ± 1. For the $b(x_i)$ term, we crudely bound the derivative, take absolute values and then use the inequality $\alpha\beta \leq \frac{1}{2}(\alpha^2+\beta^2)$

$$\begin{aligned}&\sum_{x_i} 2w^{(2n)}(t,\mathbf{x})e^{-\gamma^{(n)}|x_i|}b(x_i)\nabla^{(1)}_{x_i}w^{(2n)}(t,\mathbf{x}) \\ &\quad\leq M\sum_{x_i}|w^{(2n)}(t,\mathbf{x})|e^{-\gamma^{(n)}|x_i|}\left(|w^{(2n)}(t,\mathbf{x}^{i+})|+|w^{(2n)}(t,\mathbf{x}^{i-})|\right) \\ &\quad\leq \frac{M}{2}\sum_{x_i} e^{-\gamma^{(n)}|x_i|}\left(2(w^{(2n)}(t,\mathbf{x}))^2+(w^{(2n)}(t,\mathbf{x}^{i+}))^2+(w^{(2n)}(t,\mathbf{x}^{i-}))^2\right) \\ &\quad= \frac{M}{2}\sum_{x_i}(w^{(2n)}(t,\mathbf{x}))^2\left(2e^{-\gamma^{(n)}|x_i|}+e^{-\gamma^{(n)}|x_i-1|}+e^{-\gamma^{(n)}|x_i+1|}\right) \\ &\qquad+\frac{M}{2}\left((w^{(2n)}(t,\mathbf{x}))^2|_{x_i=x_{i+1}-1}e^{-\gamma^{(n)}|x_{i+1}-1|}-(w^{(2n)}(t,\mathbf{x}))^2|_{x_i=x_{i-1}}e^{-\gamma^{(n)}|x_{i-1}|}\right) \\ &\qquad+\frac{M}{2}\left((w^{(2n)}(t,\mathbf{x}))^2|_{x_i=x_{i-1}+1}e^{-\gamma^{(n)}|x_{i-1}+1|}-(w^{(2n)}(t,\mathbf{x}))^2|_{x_i=x_{i+1}}e^{-\gamma^{(n)}|x_{i+1}|}\right),\end{aligned}$$

where the last equality holds by changing variables, leading to additional boundary terms (a subset of which appear in the cases $i=1$ and $2n$). The boundary conditions for $w^{(2n)}(t,\mathbf{x})$ give $(w^{(2n)}(t,\mathbf{x}))^2|_{x_i=x_{i-1}} = (w^{(2n)}(t,\mathbf{x}))^2|_{x_i=x_{i+1}} = 0$. Replacing the remaining boundary terms with the full sum $\sum_{x_i}$ we arrive at the bound

$$\begin{aligned}&\sum_{x_i} 2w^{(2n)}(t,\mathbf{x})e^{-\gamma^{(n)}|x_i|}b(x_i)\nabla^{(1)}_{x_i}w^{(2n)}(t,\mathbf{x}) \\ &\qquad\qquad\leq M\sum_{x_i}(w^{(2n)}(t,\mathbf{x}))^2\left(e^{-\gamma^{(n)}|x_i|}+e^{-\gamma^{(n)}|x_i-1|}+e^{-\gamma^{(n)}|x_i+1|}\right).\end{aligned}$$

Noting that $e^{-\gamma^{(n)}(|x\pm 1|-|x|)} \leq e^{\gamma^{(n)}}$, we have the bound $e^{-\gamma^{(n)}|x-1|}+e^{-\gamma^{(n)}|x+1|} \leq$

$2e^{\gamma^{(n)}}e^{-\gamma^{(n)}|x|}$. All together

$$\sum_{x_i} 2w^{(2n)}(t,\mathbf{x})e^{-\gamma^{(n)}|x_i|}b(x_i)\nabla_{x_i}^{(1)}w^{(2n)}(t,\mathbf{x}) \\ \leq M(1+2e^{\gamma^{(n)}})\sum_{x_i}(w^{(2n)}(t,\mathbf{x}))^2e^{-\gamma^{(n)}|x_i|}.$$

Crudely bounding the discrete Laplacian, the above also gives a bound for the $a(x_i)$ term

$$\begin{aligned}
&\sum_{x_i} 2w^{(2n)}(t,\mathbf{x})e^{-\gamma^{(n)}|x_i|}a(x_i)\Delta_{x_i}^{(1)}w^{(2n)}(t,\mathbf{x}) \\
&\quad\leq 2M\sum_{x_i}|w^{(2n)}(t,\mathbf{x})|e^{-\gamma^{(n)}|x_i|}\left(|w^{(2n)}(t,\mathbf{x}^{i+})|+|w^{(2n)}(t,\mathbf{x}^{i-})|+2|w^{(2n)}(t,\mathbf{x})|\right) \\
&\quad\leq 2M\sum_{x_i}(w^{(2n)}(t,\mathbf{x}))^2\left(3e^{-\gamma^{(n)}|x_i|}+e^{-\gamma^{(n)}|x_i-1|}+e^{-\gamma^{(n)}|x_i+1|}\right) \\
&\quad\leq 2M(3+2e^{\gamma^{(n)}})\sum_{x_i}(w^{(2n)}(t,\mathbf{x}))^2e^{-\gamma^{(n)}|x_i|}.
\end{aligned}$$

Bringing everything together, we arrive at

$$\frac{\partial}{\partial t}E(t)\leq M(7+6e^{\gamma^{(n)}})E(t).$$

Noting that $E(0)=0$, integrating over $[0,t]$ gives

$$E(t)\leq M(7+6e^{\gamma^{(n)}})\int_0^t E(s)\,\mathrm{d}s.$$

Finally, Gronwall's inequality [18] gives the desired equality $E(t)=0$ for all $t\in[0,\infty)$. □

Appendix B

Convergence of whole space PDE approximations

We establish a sufficient condition for uniform convergence of lattice approximations to the heat equation on $\mathbb{R}^2$, along with first and second derivatives. The condition requires suitable convergence of the initial conditions.

Let $(u_t(\mathbf{x}) : t \geq 0, \mathbf{x} = (x, y) \in \mathbb{R}^2)$ be the solution to the heat equation

$$\partial_t u_t(x, y) = \Delta u_t(x, y) \quad \text{for } t \in [0, \infty) \text{ and } x,\, y \in \mathbb{R}^2, \tag{B.1}$$

with initial condition $u_0 : \mathbb{R}^2 \to \mathbb{R}$. The aim is to show that u_t is, together with its derivatives, close to the corresponding discrete equation

$$\partial_t v_t(x, y) = \Delta^{(\epsilon)} v_t(x, y) \quad \text{for } t \in [0, \infty) \text{ and } x,\, y \in \epsilon\mathbb{Z}^2, \tag{B.2}$$

with initial condition $v_0 : \epsilon\mathbb{Z}^2 \to \mathbb{R}$ a suitable approximation to u_0, where the central approximation to the Laplacian is given by (4.7) and $\Delta^{(\epsilon)} f(\mathbf{x}) = (\Delta_x^{(\epsilon)} + \Delta_y^{(\epsilon)}) f(\mathbf{x})$ for $f : \epsilon\mathbb{Z}^2 \to \mathbb{R}$. Closeness is measured in the supremum norm, given by $\|f\|_{l_\infty} = \sup_{\mathbf{x} \in \epsilon\mathbb{Z}^2} |f(\mathbf{x})|$. We write D^α for a mixed derivative with multi-index $\alpha \in \mathbb{N}^2$. Noting that the kernels of chapter 3 involve right discrete derivatives, we write D_ϵ^α for the discrete counterpart with mixed right derivatives. Convolution on $\epsilon\mathbb{Z}^2$ is given by (4.20). The key estimates are given in the following lemma.

Lemma 12. *Fix $t > 0$, $\epsilon > 0$ and $\delta = \epsilon^k$ for some $k < 2$. Then the solutions to* (B.1) *and* (B.2)*, with bounded initial conditions u_0 and v_0, satisfy*

$$\|D^\alpha u_t - D_\epsilon^\alpha v_t\|_{l_\infty} \leq C(\epsilon, t)\|u_0\|_{L_\infty} + C(t)\|(v_0 - P_\delta u_0) \star p_{t/2}^{(\epsilon)}\|_{l_\infty}, \tag{B.3}$$

for multi-indices $\alpha \in \mathbb{N}^2$ with $|\alpha| \leq 2$, where $C(\epsilon, t) \to 0$ as $\epsilon \downarrow 0$, $C(t)$ is independent of ϵ, P_t is the heat equation semigroup, and $p_t^{(\epsilon)} : \epsilon\mathbb{Z}^2 \to \mathbb{R}$ is the discrete heat kernel on $\epsilon\mathbb{Z}^2$.

The discrete heat kernel $p_t^{(\epsilon)} : \epsilon\mathbb{Z}^2 \to \mathbb{R}$ on $\epsilon\mathbb{Z}^2$ satisfies

$$\begin{cases} \partial_t p_t^{(\epsilon)}(x,y) &= \Delta^{(\epsilon)} p_t^{(\epsilon)}(x,y) & \text{for } x,y \in \epsilon\mathbb{Z},\ t>0, \\ p_0^{(\epsilon)}(x,y) &= 1(x=0)1(y=0) & \text{for } x,y \in \epsilon\mathbb{Z}, \end{cases}$$

and we have the explicit convolution expression $v_t(x,y) = (v_0 \star p_t^{(\epsilon)})(x,y)$ for the solution to the discrete heat equation (B.2) with initial condition $v_0(x,y)$. Note that due to the central Laplacian approximation and symmetric initial condition, the discrete heat kernel satisfies the symmetry condition $p_t^{(\epsilon)}(x,y) = p_t^{(\epsilon)}(-x,-y)$. The solution to (B.1) with initial condition $u_0(x,y)$ may be written in terms of the semigroup P_t as $u_t(x,y) = P_t u_0(x,y)$. Note that in general $\|v_0 - P_\delta u_0\|_{l_\infty}$ does not vanish as $\epsilon \downarrow 0$ because the initial conditions v_0 may be highly oscillatory. However one expects the right-hand side of (B.3) is small exactly when v_0 is a reasonable 'distributional' approximation to u_0. Rephrasing the lemma, we arrive at the following sufficient conditions for uniform convergence of heat equation approximations.

Lemma 13. *For fixed $t > 0$, the solutions at time t of* (B.1) *and* (B.2)*, with bounded initial conditions u_0 and v_0, along with their first and second derivatives, converge uniformly provided*

$$\|(v_0^{(i)} - P_\delta u_0^{(i)}) \star p_{t/2}^{(\epsilon)}\|_{l_\infty} \to 0 \qquad \text{as } \epsilon \downarrow 0,$$

where $\delta = \epsilon^k$ for some $k < 2$ and P_t, $p_t^{(\epsilon)}$ are as in lemma 12.

Remark 29. The PDE estimates underlying the sufficient condition are fairly standard but there are several subtleties that take it outside the standard theory. For example, the domains are unbounded, the solutions are bounded but not necessarily decaying, and the initial conditions only converge in distribution. We provide details of the method here, since we found it hard to find a clean account.

Before embarking on the proof, we collect some estimates for derivatives of u_t and v_t. Throughout $C \geq 0$ may vary from line to line, but its dependence on parameters will always be indicated. In particular, $C(\epsilon, t)$ always denotes a quantity converging to zero as $\epsilon \downarrow 0$. We begin with well-known heat equation estimates for u_t. The L_1 and L_∞ norms on $\mathbb{R}^2$ are given, for $f : \mathbb{R}^2 \to \mathbb{R}$, by $\|f\|_{L_1} = \int_{\mathbb{R}^2} |f(\mathbf{x})|\, d\mathbf{x}$ and $\|f\|_{L_\infty} = \sup_{\mathbf{x}\in\mathbb{R}^2} |f(\mathbf{x})|$.

Lemma 14. *For all $t > 0$ and multi-indices $\alpha \in \mathbb{N}^2$*

1. $\|D^\alpha u_t\|_{L_\infty} \leq Ct^{-\frac{|\alpha|}{2}}\|u_0\|_{L_\infty}$,
2. $\|D^\alpha u_t\|_{L_1} \leq Ct^{-\frac{|\alpha|}{2}}\|u_0\|_{L_1}$.

The above estimates may be proved by exploiting the convolution formula for u_t with the explicit Green's function. An alternative method is to use Fourier transforms, which has the advantage of a tractable discrete analogue. The l_1 norm and inner product are given, for $f, g : \epsilon\mathbb{Z}^2 \to \mathbb{R}$, by

$$\|f\|_{l_1} = \epsilon^2 \sum_{\mathbf{x}\in\mathbb{Z}^2} |f(\epsilon\mathbf{x})|, \qquad \langle f, g\rangle = \epsilon^2 \sum_{\mathbf{x}\in\mathbb{Z}^2} f(\epsilon\mathbf{x})g(\epsilon\mathbf{x}).$$

Note that for small times discrete effects kick in.

Lemma 15. *For all $\epsilon > 0$ and $t > 0$*

$$\|v_t\|_{l_1} \leq \|v_0\|_{l_1}.$$

For all $\epsilon > 0$, $t \geq \epsilon^2$ and multi-indices $\alpha \in \mathbb{N}^2$

1. $\|D^\alpha_\epsilon v_t\|_{l_\infty} \leq Ct^{-\frac{|\alpha|}{2}}\|v_0\|_{l_\infty}$,
2. $\|D^\alpha_\epsilon v_t\|_{l_1} \leq Ct^{-\frac{|\alpha|}{2}}\|v_0\|_{l_1}$.

The remaining estimates are consequences of these basic results. Firstly, the analogous estimates hold for the central discrete approximation to the Laplacian.

Corollary 2. *For all $\epsilon > 0$ and $t \geq \epsilon^2$*

1. $\|\Delta^{(\epsilon)} v_t\|_{l_\infty} \leq Ct^{-1}\|v_0\|_{l_\infty}$.
2. $\|\Delta^{(\epsilon)} v_t\|_{l_1} \leq Ct^{-1}\|v_0\|_{l_1}$.

The mean value theorem gives error estimates between first and second order derivatives of u_t and their discrete analogues.

Corollary 3. *For all $\epsilon > 0$ and $t > 0$, the central discrete derivative satisfies*

$$\|\Delta u_t - \Delta^{(\epsilon)}_x u_t\|_{L_\infty} \leq C\epsilon t^{-\frac{3}{2}}\|u_0\|_{L_\infty}.$$

Similarly, for right discrete derivatives and $|\alpha| \leq 2$

$$\|D^\alpha u_t - D^\alpha_\epsilon u_t\|_{L_\infty} \leq C\epsilon t^{-\frac{|\alpha|+1}{2}}\|u_0\|_{L_\infty}.$$

The mean value theorem also gives bounds for combinations of derivatives.

Corollary 4. *For all $\epsilon > 0$, $t > 0$ and multi-indices $\alpha, \beta \in \mathbb{N}^2$*

1. $\|D^\beta D_\epsilon^\alpha u_t\|_{L_\infty} \leq Ct^{-\frac{|\alpha|+|\beta|}{2}} \|u_0\|_{L_\infty}$.
2. $\|\Delta^{(\epsilon)} D_\epsilon^\alpha u_t\|_{L_\infty} \leq Ct^{-\frac{|\alpha|+2}{2}} \|u_0\|_{L_\infty}$.

Corollary 5. *For all $\epsilon > 0$, $t \geq \epsilon^2$ and multi-indices $\alpha \in \mathbb{N}^2$*

$$\|\Delta^{(\epsilon)} D_\epsilon^\alpha v_t\|_{l_1} \leq Ct^{-\frac{|\alpha|+2}{2}} \|v_0\|_{l_1}.$$

With these results in hand, we now turn to the proof of lemma 12.

Proof of lemma 12. We begin with the proof of (B.3) for $|\alpha| = 0$, which forms the template for the $|\alpha| = 1, 2$ cases. The supremum norm of a function may be controlled by uniformly bounding the inner product against test functions, as shown in the following lemma.

Lemma 16. *Let $f : \epsilon\mathbb{Z}^2 \to \mathbb{R}$ satisfy $|\langle f, \psi\rangle| \leq C\|\psi\|_{l_1}$ for all $\psi : \epsilon\mathbb{Z}^2 \to \mathbb{R}$ and some $C > 0$. Then $\|f\|_{l_\infty} \leq C$.*

Proof of lemma 16. We prove the contrapositive. With this in mind, suppose that $\|f\|_{l_\infty} > C$, then there exists $\mathbf{y} \in \epsilon\mathbb{Z}^2$ such that $f(\mathbf{y}) > C$. Defining $\psi : \epsilon\mathbb{Z}^2 \to \mathbb{R}$ to be the indicator of the set $\{\mathbf{y}\}$

$$|\langle f, \psi\rangle| = \epsilon^2 f(\mathbf{y}) > \epsilon^2 C = C\|\psi\|_{l_1},$$

and the contrapositive is established. □

Setting $w_t = u_t - v_t$, it suffices to show

$$|\langle w_t, \psi_0\rangle| \leq \left(C(\epsilon, t)\|u_0\|_{L_\infty} + C(t)\|(v_0 - P_\delta u_0) \star p_{t/2}^{(\epsilon)}\|_{l_\infty}\right) \|\psi_0\|_{l_1}.$$

Here ψ_0 forms the initial condition of $(\psi_t(\mathbf{x}) : t \geq 0, \mathbf{x} \in \epsilon\mathbb{Z}^2)$, also solving (B.2)

$$\partial_t \psi_t(\mathbf{x}) = \Delta^{(\epsilon)} \psi_t(\mathbf{x}) \quad \text{on } [0, \infty) \times \epsilon\mathbb{Z}^2. \tag{B.4}$$

The reason for taking this equation is that we may develop the inner product as follows

$$\langle w_t, \psi_0\rangle = \langle w_0, \psi_t\rangle + \int_0^t \langle \partial_s w_s, \psi_{t-s}\rangle + \langle w_s, \partial_s \psi_{t-s}\rangle \, \mathrm{d}s.$$

Using (B.4) we may replace the time derivative by a discrete spatial derivative $\Delta^{(\epsilon)}$, then move this operator from ψ_{t-s} onto w_s by taking its adjoint. In order to cancel and combine terms of the integrand, the natural choice for ψ_t is the operator whose adjoint is (B.2), namely itself. Note that the estimates in lemma 15 and corollaries 2 and 5 hold for ψ_t. Following the outline, for the v_t component of w_t

$$\begin{aligned}\langle v_t, \psi_0\rangle &= \langle v_0, \psi_t\rangle + \int_0^t \langle \partial_s v_s, \psi_{t-s}\rangle + \langle v_s, \partial_s \psi_{t-s}\rangle \,\mathrm{d}s \\ &= \langle v_0, \psi_t\rangle + \int_0^t \left(\langle \Delta^{(\epsilon)} v_s, \psi_{t-s}\rangle - \langle v_s, \Delta^{(\epsilon)}\psi_{t-s}\rangle\right) \mathrm{d}s. \end{aligned} \tag{B.5}$$

The adjoint remarks above may be formalised by expanding the discrete operators and recombining, giving the following summation by parts formula

$$\begin{aligned}\langle v_s, \Delta^{(\epsilon)}\psi_{t-s}\rangle &= \sum_{i=1}^{2}\sum_{\mathbf{x}\in\mathbb{Z}^2} v_s(\epsilon\mathbf{x})\left(\psi_{t-s}(\epsilon\mathbf{x}+\epsilon e_i) + \psi_{t-s}(\epsilon\mathbf{x}-\epsilon e_i) - 2\psi_{t-s}(\epsilon\mathbf{x})\right) \\ &= \sum_{i=1}^{2}\sum_{\mathbf{x}\in\mathbb{Z}^2} \left(v_s(\epsilon\mathbf{x}-\epsilon e_i) + v_s(\epsilon\mathbf{x}+\epsilon e_i) - 2v_s(\epsilon\mathbf{x})\right)\psi_{t-s}(\epsilon\mathbf{x}) \\ &= \langle \Delta^{(\epsilon)} v_s, \psi_{t-s}\rangle. \end{aligned} \tag{B.6}$$

Substituting back into (B.5) and cancelling terms we arrive at

$$\langle v_t, \psi_0\rangle = \langle v_0, \psi_t\rangle.$$

For $0 < \delta < t$, we may similarly develop the u_t term using summation by parts

$$\begin{aligned}\langle u_t, \psi_0\rangle &= \langle u_\delta, \psi_{t-\delta}\rangle + \int_\delta^t \langle \partial_s w_s, \psi_{t-s}\rangle + \langle w_s, \partial_s\psi_{t-s}\rangle \,\mathrm{d}s \\ &= \langle u_\delta, \psi_{t-\delta}\rangle + \int_\delta^t \left(\langle \Delta u_s, \psi_{t-s}\rangle - \langle u_s, \Delta^{(\epsilon)}\psi_{t-s}\rangle\right) \mathrm{d}s \\ &= \langle u_\delta, \psi_{t-\delta}\rangle + \int_\delta^t \langle E(u_s), \psi_{t-s}\rangle \,\mathrm{d}s, \end{aligned}$$

where $E(u_s) = \Delta u_s - \Delta^{(\epsilon)} u_s$. Using corollary 3

$$\|E(u_s)\|_{L_\infty} = \|\Delta u_s - \Delta^{(\epsilon)} u_s\|_{L_\infty} \leq C\epsilon s^{-\frac{3}{2}} \|u_0\|_{L_\infty}.$$

Note that it is these estimates that prevent the choice $\delta = 0$. We implicitly assume that ϵ is sufficiently small with respect to t, so we can apply lemma 15 provided

$\delta \geq \epsilon^2$

$$\begin{aligned}\Big| \int_\delta^t \langle E(u_s), \psi_{t-s} \rangle \, \mathrm{d}s \Big| &\leq C\epsilon \|u_0\|_{L_\infty} \int_\delta^t s^{-\frac{3}{2}} \|\psi_{t-s}\|_{l_1} \, \mathrm{d}s \\ &\leq C\epsilon \|u_0\|_{L_\infty} \|\psi_0\|_{l_1} \int_\delta^t s^{-\frac{3}{2}} \, \mathrm{d}s \\ &\leq C\epsilon \|u_0\|_{L_\infty} \|\psi_0\|_{l_1} \delta^{-\frac{1}{2}}.\end{aligned} \tag{B.7}$$

Setting $\delta = \epsilon^k > \epsilon^2$ for $k < 2$, the right-hand sides above converges to zero as $\epsilon \downarrow 0$, and we arrive at the estimate

$$\Big| \int_\delta^t \langle E(u_s), \psi_{t-s} \rangle \, \mathrm{d}s \Big| \leq C(\epsilon, t) \|u_0\|_{L_\infty} \|\psi_0\|_{l_1}.$$

It remains to show that $\langle u_\delta, \psi_{t-\delta} \rangle \approx \langle v_0, \psi_t \rangle$. First, the triangle inequality gives

$$|\langle u_\delta, \psi_{t-\delta} \rangle - \langle v_0, \psi_t \rangle| \leq |\langle u_\delta, \psi_t - \psi_{t-\delta} \rangle| + |\langle v_0 - u_\delta, \psi_t \rangle|. \tag{B.8}$$

For the first term, there is $s \in (t-\delta, t)$ by the mean value theorem such that

$$\langle u_\delta, \psi_t - \psi_{t-\delta} \rangle = \delta \langle u_\delta, \partial_s \psi_s \rangle = \delta \langle u_\delta, \Delta^{(\epsilon)} \psi_s \rangle.$$

Using corollary 2 and lemma 14, we obtain

$$\begin{aligned}|\langle u_\delta, \psi_t - \psi_{t-\epsilon} \rangle| &\leq \delta \|u_\delta\|_{l_\infty} \|\Delta^{(\epsilon)} \psi_s\|_{l_1} \\ &\leq C\delta (t-\delta)^{-1} \|u_0\|_{L_\infty} \|\psi_0\|_{l_1} \\ &\leq C(\epsilon, t) \|u_0\|_{L_\infty} \|\psi_0\|_{l_1}.\end{aligned}$$

For the second term of (B.8), expanding notation $u_t = P_t u_0$ and $\psi_t = \psi_0 \star p_t^{(\epsilon)}$, exchanging the inner product and convolution summations, and using the symmetry of $p_t^{(\epsilon)}$, we find

$$\langle v_0 - u_\delta, \psi_t \rangle = \langle (v_0 - P_\delta u_0) \star p_t^{(\epsilon)}, \psi_0 \rangle.$$

Note that $(v_0 - P_\delta u_0) \star p_t^{(\epsilon)}$ solves (B.2) on $[0, t]$ with initial condition $(v_0 - P_\delta u_0)$. Interpreting this as the solution on $[t/2, t]$ with initial condition $(v_0 - P_\delta u_0) \star p_{t/2}^{(\epsilon)}$ and applying lemma 15

$$\|(v_0 - P_\delta u_0) \star p_t^{(\epsilon)}\|_{l_\infty} \leq \|(v_0 - P_\delta u_0) \star p_{t/2}^{(\epsilon)}\|_{l_\infty}.$$

Combining everything, we arrive at the desired estimate

$$\begin{aligned}|\langle w_t, \psi_0\rangle| &\leq |\langle u_\delta, \psi_{t-\delta}\rangle - \langle v_0, \psi_t\rangle| + \Big| \int_\epsilon^t \langle E(u_s), \psi_{t-s}\rangle \, \mathrm{d}s\Big| \\ &\leq \Big(C(\epsilon,t)\|u_0\|_{L_\infty} + C\|(v_0 - P_\delta u_0) \star p^{(\epsilon)}_{t/2}\|_{l_\infty}\Big) \, \|\psi_0\|_{l_1}.\end{aligned}$$

We now turn to the approximation of derivatives. With lemma 16 in mind, the aim is to control $|\langle D^\alpha_\epsilon w_t, \psi_0\rangle|$. The outline is to repeat the above steps by moving the discrete derivative onto ψ_0. For the central approximation $\Delta^{(\epsilon)}$, we have seen the summation by parts formula (B.6) for moving derivatives. However, for right derivatives D^α_ϵ, the adjoint operator is given by left derivatives. Indeed the one-dimensional case is given, for $f, g : \epsilon\mathbb{Z} \to \mathbb{R}$ with operators D^+ and D^- defined by (3.5), by

$$\begin{aligned}\langle D^+ f, g\rangle &= \epsilon \sum_{x\in\mathbb{Z}} \left(f(\epsilon x + \epsilon) - f(\epsilon x)\right) g(\epsilon x) \\ &= \epsilon \sum_{x\in\mathbb{Z}} f(\epsilon x) \left(g(\epsilon x - \epsilon) - g(\epsilon x)\right) = \langle f, D^- g\rangle.\end{aligned}$$

Denoting multi-index left derivatives by $\tilde{D}^\alpha_\epsilon$, the general formula for $f, g : \epsilon\mathbb{Z}^2 \to \mathbb{R}$ is

$$\langle D^\alpha_\epsilon f, g\rangle = \langle f, \tilde{D}^\alpha_\epsilon g\rangle.$$

Thus, we may control $|\langle D^\alpha_\epsilon w_t, \psi_0\rangle|$ by developing $|\langle w_t, \tilde{D}^\alpha_\epsilon \psi_0\rangle|$. We derive the following bound

$$|\langle w_t, \tilde{D}^\alpha_\epsilon \psi_0\rangle| \leq \Big(C(\epsilon,t)\|u_0\|_{L_\infty} + C(t)\|(v_0 - P_\delta u_0) \star p^{(\epsilon)}_{t/2}\|_{l_\infty}\Big) \, \|\psi_0\|_{l_1}. \tag{B.9}$$

This implies that

$$\|D^\alpha_\epsilon u_t - D^\alpha_\epsilon v_t\|_{l_\infty} \leq C(\epsilon,t)\|u_0\|_{L_\infty} + C(t)\|(v_0 - P_\delta u_0) \star p^{(\epsilon)}_{t/2}\|_{l_\infty},$$

and corollary 3 gives $\|D^\alpha u_t - D^\alpha_\epsilon u_t\|_{l_\infty} \leq C(\epsilon,t)\|u_0\|_{L_\infty}$, finishing the uniform convergence of derivatives. The calculation for the v_t term goes through with $\tilde{D}^\alpha_\epsilon \psi_0$. However, to establish (B.9) there is a small change for the u_t term, since analogues of estimates like (B.7) diverge for $|\alpha| \geq 2$, namely

$$\int_\delta^t s^{-\frac{3}{2}} \|\tilde{D}^\alpha_\epsilon \psi_{t-s}\|_{l_1} \, \mathrm{d}s \leq \|\psi_0\|_{l_1} \int_\delta^t s^{-\frac{3}{2}} (t-s)^{-\frac{|\alpha|}{2}} \, \mathrm{d}s.$$

For $|\alpha| = 1$ we could get away without the modification, but it is convenient to unify the proof for the cases $|\alpha| = 1$, 2. The modification is to fix $0 < \delta < t/2$ and argue up to time $t - \delta$ as follows

$$\langle u_{t-\delta}, \tilde{D}_\epsilon^\alpha \psi_\delta \rangle = \langle u_\delta, \tilde{D}_\epsilon^\alpha \psi_{t-\delta} \rangle + \int_\delta^{\frac{t}{2}} \langle E(u_s), \tilde{D}_\epsilon^\alpha \psi_{t-s} \rangle \, \mathrm{d}s + \int_{\frac{t}{2}}^{t-\delta} \langle E(u_s), \tilde{D}_\epsilon^\alpha \psi_{t-s} \rangle \, \mathrm{d}s.$$

The first integral on the right-hand side may be treated as before and we show that the second may also be bounded. We then show that $\langle u_\delta, \tilde{D}_\epsilon^\alpha \psi_{t-\delta} \rangle \approx \langle v_0, \tilde{D}_\epsilon^\alpha \psi_{t-\delta} \rangle$, similarly to before, but there is an extra step to show that $\langle u_t, \tilde{D}_\epsilon^\alpha \psi_0 \rangle \approx \langle u_{t-\delta}, \tilde{D}_\epsilon^\alpha \psi_\delta \rangle$. All together we split the desired quantity as follows

$$\begin{aligned}
|\langle w_t, \tilde{D}_\epsilon^\alpha \psi_0 \rangle| &= |\langle u_t, \tilde{D}_\epsilon^\alpha \psi_0 \rangle - \langle v_0, \tilde{D}_\epsilon^\alpha \psi_t \rangle| \\
&\leq |\langle u_t, \tilde{D}_\epsilon^\alpha \psi_0 \rangle - \langle u_{t-\delta}, \tilde{D}_\epsilon^\alpha \psi_\delta \rangle| + |\langle u_{t-\delta}, \tilde{D}_\epsilon^\alpha \psi_\delta \rangle - \langle v_0, \tilde{D}_\epsilon^\alpha \psi_t \rangle| \\
&\leq |\langle u_t, \tilde{D}_\epsilon^\alpha \psi_0 \rangle - \langle u_{t-\delta}, \tilde{D}_\epsilon^\alpha \psi_\delta \rangle| + |\langle u_\delta, \tilde{D}_\epsilon^\alpha \psi_{t-\delta} \rangle - \langle v_0, \tilde{D}_\epsilon^\alpha \psi_t \rangle| \\
&\quad + \Big| \int_\delta^{\frac{t}{2}} \langle E(u_s), \tilde{D}_\epsilon^\alpha \psi_{t-s} \rangle \, \mathrm{d}s \Big| + \Big| \int_{\frac{t}{2}}^{t-\delta} \langle E(u_s), \tilde{D}_\epsilon^\alpha \psi_{t-s} \rangle \, \mathrm{d}s \Big|. \qquad \text{(B.10)}
\end{aligned}$$

Noting that lemma 15 holds unchanged for left derivatives, we may bound the $\delta < s < \frac{t}{2}$ error term analogously to the $|\alpha| = 0$ case, bounding $\|\tilde{D}_\epsilon^\alpha \psi_{t-s}\|_{l_1}$ by the worst case $s = \frac{t}{2}$

$$\begin{aligned}
\Big| \textstyle\int_\delta^{\frac{t}{2}} \langle E(u_s), \tilde{D}_\epsilon^\alpha \psi_{t-s} \rangle \, \mathrm{d}s \Big| &\leq C\epsilon \|u_0\|_{L_\infty} \|\psi_0\|_{l_1} \textstyle\int_\delta^{\frac{t}{2}} s^{-\frac{3}{2}} (t-s)^{-\frac{|\alpha|}{2}} \, \mathrm{d}s \\
&\leq C\epsilon \|u_0\|_{L_\infty} \|\psi_0\|_{l_1} \delta^{-\frac{1}{2}} \left(\tfrac{t}{2}\right)^{-\frac{|\alpha|}{2}}.
\end{aligned}$$

For $\frac{t}{2} < s < t - \delta$ we must take care with the $\tilde{D}_\epsilon^\alpha \psi_{t-s}$ term but can bound the rest of the integrands by their largest values

$$\begin{aligned}
\Big| \int_{\frac{t}{2}}^{t-\delta} \langle E(u_s), \tilde{D}_\epsilon^\alpha \psi_{t-s} \rangle \, \mathrm{d}s \Big| &\leq C\epsilon \|u_0\|_{L_\infty} \|\psi_0\|_{l_1} \int_{\frac{t}{2}}^{t-\delta} s^{-\frac{3}{2}} (t-s)^{-\frac{|\alpha|}{2}} \, \mathrm{d}s \\
&\leq C\epsilon \|u_0\|_{L_\infty} \|\psi_0\|_{l_1} \left(\frac{t}{2}\right)^{-\frac{3}{2}} f_\alpha\left(\frac{t}{2}, t-\delta\right),
\end{aligned}$$

where

$$f_\alpha\left(\frac{t}{2}, t-\delta\right) = \int_{\frac{t}{2}}^{t-\delta} (t-s)^{-\frac{|\alpha|}{2}} \, \mathrm{d}s \leq \begin{cases} 2\left(\frac{t}{2}\right)^{\frac{1}{2}} & \text{if } |\alpha| = 1, \\ \ln\left(\frac{t}{2\delta}\right) & \text{if } |\alpha| = 2. \end{cases}$$

Since $\delta = \epsilon^k > \epsilon^2$ for $k < 2$, both error bounds converge to zero as $\epsilon \downarrow 0$, giving

$$\Big| \int_{\delta}^{\frac{t}{2}} \langle E(u_s), \tilde{D}_\epsilon^\alpha \psi_{t-s} \rangle \, \mathrm{d}s \Big| + \Big| \int_{\frac{t}{2}}^{t-\delta} \langle E(u_s), \tilde{D}_\epsilon^\alpha \psi_{t-s} \rangle \, \mathrm{d}s \Big| \leq C(\epsilon, t) \|u_0\|_{L_\infty} \|\psi_0\|_{l_1}.$$

We turn to the remaining error estimates of (B.10). Firstly consider

$$|\langle u_\delta, \tilde{D}_\epsilon^\alpha \psi_{t-\delta} \rangle - \langle v_0, \tilde{D}_\epsilon^\alpha \psi_t \rangle| \leq |\langle u_\delta, \tilde{D}_\epsilon^\alpha (\psi_t - \psi_{t-\delta}) \rangle| + |\langle v_0 - u_\delta, \tilde{D}_\epsilon^\alpha \psi_t \rangle|. \quad \text{(B.11)}$$

Following the same steps as before, for the first term there is $s \in (t-\delta, t)$, by the mean value theorem, satisfying

$$\begin{aligned}
|\langle u_\delta, \tilde{D}_\epsilon^\alpha (\psi_t - \psi_{t-\delta}) \rangle| &= \delta |\langle u_\delta, \tilde{D}_\epsilon^\alpha \Delta^{(\epsilon)} \psi_s \rangle| \\
&\leq \delta \|u_\delta\|_{l_\infty} \|\tilde{D}_\epsilon^\alpha \Delta^{(\epsilon)} \psi_s\|_{l_1} \\
&\leq C\delta (t-\delta)^{-\frac{|\alpha|+2}{2}} \|u_0\|_{l_\infty} \|\psi_0\|_{l_1} \\
&\leq C(\epsilon, t) \|u_0\|_{l_\infty} \|\psi_0\|_{l_1},
\end{aligned}$$

where we use corollary 5 replacing right with left derivatives. Moving the discrete derivative inside the convolution, we develop the second term of (B.11)

$$\begin{aligned}
|\langle v_0 - u_\delta, \tilde{D}_\epsilon^\alpha \psi_t \rangle| &= |\langle v_0 - P_\delta u_0, (\tilde{D}_\epsilon^\alpha \psi_0) \star p_t^{(\epsilon)} \rangle| \\
&= |\langle (v_0 - P_\delta u_0) \star p_t^{(\epsilon)}, \tilde{D}_\epsilon^\alpha \psi_0 \rangle| \\
&= |\langle D_\epsilon^\alpha \left((v_0 - P_\delta u_0) \star p_t^{(\epsilon)} \right), \psi_0 \rangle| \\
&\leq \|D_\epsilon^\alpha \left((v_0 - P_\delta u_0) \star p_t^{(\epsilon)} \right) \|_{l_\infty} \|\psi_0\|_{l_1}.
\end{aligned}$$

As before, interpreting $(v_0 - P_\delta u_0) \star p_t^{(\epsilon)}$ as the solution to (B.2) on $[t/2, t]$ with initial condition $(v_0 - P_\delta u_0) \star p_{t/2}^{(\epsilon)}$ and applying lemma 15

$$\|D_\epsilon^\alpha \left((v_0 - P_\delta u_0) \star p_t^{(\epsilon)} \right) \|_{l_\infty} \leq C t^{-\frac{|\alpha|}{2}} \|(v_0 - P_\delta u_0) \star p_{t/2}^{(\epsilon)}\|_{l_\infty}.$$

All together, we obtain the following bound on (B.11)

$$|\langle u_\delta, \tilde{D}_\epsilon^\alpha \psi_{t-\delta} \rangle - \langle v_0, \tilde{D}_\epsilon^\alpha \psi_{t-\delta} \rangle| \leq \Big(C(\epsilon, t) \|u_0\|_{L_\infty} + C(t) \|(v_0 - P_\delta u_0) \star p_{t/2}^{(\epsilon)}\|_{l_\infty} \Big) \|\psi_0\|_{l_1}.$$

The final error term to control is given by

$$|\langle u_t, \tilde{D}_\epsilon^\alpha \psi_0 \rangle - \langle u_{t-\delta}, \tilde{D}_\epsilon^\alpha \psi_\delta \rangle| \leq |\langle u_t, \tilde{D}_\epsilon^\alpha \psi_0 - \tilde{D}_\epsilon^\alpha \psi_\delta \rangle| + |\langle u_t - u_{t-\delta}, \tilde{D}_\epsilon^\alpha \psi_\delta \rangle|. \quad \text{(B.12)}$$

For the first term, there is $s \in (0, \delta)$ such that

$$|\langle u_t, \tilde{D}_\epsilon^\alpha \psi_0 - \tilde{D}_\epsilon^\alpha \psi_\delta \rangle| = \delta |\langle \Delta^{(\epsilon)} D_\epsilon^\alpha u_t, \psi_s \rangle| \leq \delta C t^{-\frac{|\alpha|+2}{2}} \|u_0\|_{l_\infty} \|\psi_0\|_{l_1},$$

where the inequality follows from corollary 4 and lemma 15, giving the bound $C(\epsilon, t)\|u_0\|_{l_\infty}\|\psi_0\|_{l_1}$. Similarly, for the second term of (B.12), there is $s \in (t-\delta, t)$ such that

$$|\langle u_t - u_{t-\delta}, \tilde{D}_\epsilon^\alpha \psi_\delta \rangle| = \delta |\langle D_\epsilon^\alpha \Delta u_s, \psi_\delta \rangle| \leq \delta C (t-\delta)^{-\frac{|\alpha|+2}{2}} \|u_0\|_{L_\infty} \|\psi_0\|_{l_1}.$$

All together, (B.12) is bounded

$$|\langle u_t, \tilde{D}_\epsilon^\alpha \psi_0 \rangle - \langle u_{t-\delta}, \tilde{D}_\epsilon^\alpha \psi_\delta \rangle| \leq C(\epsilon, t)\|u_0\|_{L_\infty}\|\psi_0\|_{l_1}.$$

Finally, substituting everything into (B.10), we arrive at the desired bound

$$|\langle w_t, \tilde{D}_\epsilon^\alpha \psi_0 \rangle| \leq \left(C(\epsilon, t)\|u_0\|_{L_\infty} + C(t)\|(v_0 - P_\delta u_0) \star p_{t/2}^{(\epsilon)}\|_{l_\infty} \right) \|\psi_0\|_{l_1}.$$

This completes the proof of lemma B.3. □

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